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# Hydrodynamic Limit of a B.G.K. Like Model on Domains with Boundaries and Analysis of Kinetic Boundary Conditions for Scalar Multidimensional Conservation Laws

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In this paper we study the hydrodynamic limit of a B.G.K. like kinetic model on domains with boundaries via $BV_{\text{loc}}$ theory. We obtain as a consequence existence results for scalar multidimensional conservation laws with kinetic boundary conditions. We require that the initial and boundary data satisfy the optimal assumptions that they all belong to $L^1 \cap L^\infty$ with the additional regularity assumptions that the initial data are in $BV_{\text{loc}}$. We also extend our hydrodynamic limit analysis to the case of a generalized kinetic model to account for forces effects and we obtain as a consequence the existence theory for conservation laws with source terms and kinetic boundary conditions.

**KEY WORDS:** Domains with boundaries; hydrodynamic limit; kinetic boundary conditions; scalar multidimensional conservation laws.

## 1. INTRODUCTION

In this paper we consider the following kinetic model

$$[\partial_t + a(v) \cdot \partial_x]\, g_\epsilon(x, v, t) = \frac{1}{\epsilon}\left(\chi_{w_\epsilon(x,t)}(v) - g_\epsilon(x, v, t)\right) \quad \text{in} \quad \Omega \times V \times (0, T) \tag{1}$$

$$g_\epsilon(x, v, t) = g_{\epsilon 0}(x, v, t) \quad \text{on} \quad \Gamma_0^- \times (0, T) \tag{2}$$

$$g_\epsilon(x, v, t) = g_{\epsilon 1}(x, v, t) \quad \text{on} \quad \Gamma_1^- \times (0, T), \tag{3}$$

$$g_\epsilon(x, v, 0) = g_\epsilon^0(x, v) \quad \text{in} \quad \Omega \times V \tag{4}$$

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and study its relation to the scalar multidimensional conservation laws

$$\partial_t w + \partial_{x_i}[A_i(w)] = 0 \qquad \text{in} \quad \Omega \times (0, T) \tag{5}$$

$$\text{Boundary conditions for } w \text{ on } \Gamma_0 \times (0, T) \text{ and } \Gamma_1 \times (0, T) \tag{6}$$

$$w(x, 0) = w^0(x) \qquad \text{in } \Omega \tag{7}$$

Here, $\Omega = (0, 1) \times \mathbb{R}^{d-1}$ is the physical domain. The boundaries are defined as follows

$$\Gamma_0 = \{0\} \times \mathbb{R}^{d-1}, \qquad \Gamma_1 = \{1\} \times \mathbb{R}^{d-1},$$
$$\Gamma_0^- = \{(x, v) \in \{0\} \times \mathbb{R}^{d-1} \times V : a(v) \cdot n(x) < 0\}$$
$$\Gamma_1^- = \{(x, v) \in \{1\} \times \mathbb{R}^{d-1} \times V : a(v) \cdot n(x) < 0\}$$

where $n$ denotes the exterior unit normal vector to $\Omega$. The boundary conditions in (6) for the conservation laws are prescribed on a part of $\Gamma_0$ resp. $\Gamma_1$. These boundary conditions will be precised in Definition 3.1. The set $V = \mathbb{R}$ is the velocity domain. The function $g_\epsilon$ describes the microscopic density of particles at $(x, t)$ with velocity $v$ in the kinetic domain. The function $w$ describes the local density of particles at $(x, t)$ in the hydrodynamic domain. The physical parameter $\epsilon > 0$ is the microscopic scale. The functions $g_\epsilon^0$ and $w^0$ are the initial data while $g_{\epsilon 0}$ and $g_{\epsilon 1}$ are boundary data. The boundary conditions in (6) involve also $w_1$ and $g_0$ which are given boundary data; see Definition 3.1 below. Let $A = (A_i)_{1 \leqslant i \leqslant d}$, the components of $A$ are assumed to satisfy $A_i(\cdot) \in C^1$ and are related to $a_i(\cdot)$ by $a_i(\cdot) = A_i'(\cdot)$, $i = 1, \ldots, d$. The local density of particles $w_\epsilon$ at $(x, t)$ is related to the microscopic density $g_\epsilon$ by $w_\epsilon(x, t) = \int_V g_\epsilon(x, v, t)\, dv$. The collisions in the kinetic domain are given by the nonlinear kernel in the right hand side of Eq. (1) in which $\chi_u(v)$ is the signature of $u$ defined by

$$\chi_u(v) = \begin{cases} +1 & \text{if} \quad 0 < v \leqslant u \\ -1 & \text{if} \quad u \leqslant v < 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Our main objective in this paper is to describe the conservation laws (5)–(7) as the macroscopic limit of the Boltzmann-like equations (1)–(4), as the microscopic scale, $\epsilon > 0$, goes to 0. This problem is a particular case of the more general problem of describing compressible Euler equations as the macroscopic limit of Boltzmann or B.G.K. equations, as the microscopic scale goes to 0. The convergence of the moments of the kinetic distributions of Boltzmann or B.G.K. equations to weak solutions of the compressible Euler equations is still an open problem. In the case of strong solutions this question has been solved by Caflisch.[2] The case of domains with boundaries is still completely open. In this paper we give an answer to this question in the case of the B.G.K. like model (1)–(4).

The study of the hydrodynamic limit of the kinetic model (1)–(4) in full space ($\Omega = \mathbb{R}^d$) has been initiated by Perthame and Tadmor.[7] The proofs of the main results in ref. 7 rely in a fundamental way on the finite speed of propagation together with the compactness of the support in the velocity $v$ of the microscopic density, which are in turn based on the uniform $L^\infty$ boundedness (in $\epsilon$) of the macroscopic density. But the argument used in ref. 7 to prove the $L^\infty$ uniform with respect to $\epsilon$ bound of the macroscopic density is not entirely correct (consult Remark 2.2). Later Nouri, Omrane, and Vila attempted to study this hydrodynamic limit in the case of $\mathbb{R}^+ \times \mathbb{R}^{d-1}$.[6] Unfortunately their proofs are not entirely correct. In their proofs of the various $L^\infty$, $L^1$, and $BV$ uniform, in $\epsilon$, estimates, they have used in an essential way Gronwall lemma, which does not yield the uniform bounds they claimed. These uniform bounds are central to their proofs. In ref. 6, Proposition 3 on p. 784 and Proposition 4 on p. 786, are obtained by applying Gronwall lemma to the inequality

$$V_\epsilon(t) \leqslant \int_0^t \frac{1}{\epsilon} e^{(s-t)/\epsilon} V_\epsilon(s)\, ds + C \tag{9}$$

and then they conclude that $|V_\epsilon(t)|$ is uniformly bounded. This is not the case as the following counterexample shows. Take $V_\epsilon(t) = \frac{Ct}{\epsilon}$, $V_\epsilon$ satisfies the inequality (9), however, $V_\epsilon$ is not uniformly in $\epsilon$ bounded.

In this paper, we shall see how the ideas developed by the author in refs. 14 and 15 to study the more difficult coupled system of kinetic equations (1) and their hydrodynamic limit (conservation laws of the form in (5)), which is a simplified case of the more general coupled system of Boltzmann equations and their hydrodynamic limits (compressible Euler and Navier–Stokes equations) introduced and studied in refs. 11–13 (see also the references therein), can be applied to study the hydrodynamic limit of the kinetic model (1)–(4) in the case of domains with boundaries. Our proofs rely on optimal assumptions on the initial and boundary data and do not use any technical assumptions. As a consequence of our study we are able to obtain a general proof of the $L^\infty$ uniform (in $\epsilon$) bound of the macroscopic density. This proof is valid for the case of domains with boundaries as well as the full space case. This allows us to conclude that, fortunately, the results in ref. 7 are true although their proofs given in ref. 7 are incomplete. For a further study of this problem and a generalization of the concept of kinetic formulation to conservation laws on domains with boundaries, we refer to the author's work.[16]

In the second part of this paper, we introduce a generalization of the kinetic model (1)–(4) that includes forces effects and whose macroscopic limit, as the microscopic scale go to 0, yields conservation laws with source terms. This kinetic model is more appropriate to describe the physics at the microscopic level than the model proposed in ref. 7 for the approximation of conservation laws with source terms. We then study the hydrodynamic limit of the proposed kinetic model and prove the existence theory for its continuum limit, i.e., the conservation laws with source terms.

This paper is organized as follows. In the next section we study the kinetic problem. We prove various a priori estimates that are needed for the study of the hydrodynamic limit of the kinetic problem. In Section 3, we precise our definition of physically correct solution to the problem (5)–(7). We then study the hydrodynamic limit of the kinetic problem and prove our main result. In Section 4, we study the one dimentional case via compensated compactness. We prove the convergence of the moments of the kinetic distributions to the solution of the conservation laws without any compactness argument (based on $BV_{\mathrm{loc}}$ theory). Finally in Section 5, we extend our hydrodynamic analysis to the case of a generalized kinetic model to account for forces effects and we obtain as a consequence the existence theory for conservation laws with source terms and kinetic boundary conditions.

## 2. THE KINETIC EQUATIONS

In this section we shall study various properties of the solution of the kinetic equations (1)–(4). Some of our proofs are closely related to those for the full space case in ref. 7. However, our problem is on a domain with boundaries. This introduces new difficulties that are not present in the full space case. These difficulties must be handled by different techniques. We begin by stating a result about the well posedness of the kinetic problem (1)–(4). We then establish various properties of the solution, including $L^\infty$, $L^1$, and $BV_{\mathrm{loc}}$ estimates. These estimates will be used for the study of the hydrodynamic limit of Problem (1)–(4) as $\epsilon \to 0$. We shall use the following notations.

$$\Omega_0 = \{(x, v, t) \in \Omega \times V \times (0, T) : x_1 - a_1(v)\, t < 0\}$$

$$\Omega_{01} = \{(x, v, t) \in \Omega \times V \times (0, T) : 0 < x_1 - a_1(v)\, t < 1\}$$

$$\Omega_1 = \{(x, v, t) \in \Omega \times V \times (0, T) : x_1 - a_1(v)\, t > 1\}$$

where $x = (x_1, x_\star)$.

## 2.1. Existence Theory and Basic Estimates

We establish in this section the existence and uniqueness theory and derive basic estimates for the solutions of the kinetic equations.

**Theorem 2.1.** Assume that

$$g_\epsilon^0 \in L^1(\Omega \times V),\ a(v)\cdot n g_{\epsilon 1} \in L^1(\Gamma_1^- \times (0,T)),\ a(v)\cdot n g_{\epsilon 0} \in L^1(\Gamma_0^- \times (0,T))$$

Then the problem (1)–(4) has a unique solution $g_\epsilon$ in $L^\infty((0,T); L^1(\Omega \times V))$. Moreover, $g_\epsilon$ satisfies the integral representation

In $\Omega_0$

$$g_\epsilon(x,v,t) = g_{\epsilon 0}\left(x_\star - \frac{x_1}{a_1(v)}\, a_\star(v), v, t - \frac{x_1}{a_1(v)}\right) \exp(-x_1/(a_1(v)\,\epsilon)) + \frac{1}{\epsilon}\int_{t-\frac{x_1}{a_1(v)}}^t e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s),s)}(v)\, ds$$

In $\Omega_{01}$

$$g_\epsilon(x,v,t) = g_\epsilon^0(x - a(v)\,t, v)\exp(-t/\epsilon) + \frac{1}{\epsilon}\int_0^t e^{(s-t)/\epsilon}\chi_{w_\epsilon(x(s),s)}(v)\, ds$$

In $\Omega_1$

$$g_\epsilon(x,v,t) = g_{\epsilon 1}\left(x_\star + \frac{1-x_1}{a_1(v)}\, a_\star(v), v, t - \frac{x_1 - 1}{a_1(v)}\right) \exp((1-x_1)/\epsilon a_1(v)) + \frac{1}{\epsilon}\int_{t-\frac{x_1-1}{a_1(v)}}^t e^{(s-t)/\epsilon}\chi_{w_\epsilon(x(s),s)}(v)\, ds$$

where $x(s) = x + (s-t)\, a(v)$, $x = (x_1, x_\star)$, and $a(v) = (a_1(v), a_\star(v))$.

Finally, let $g_\epsilon$ and $G_\epsilon$ be two solutions of (1)–(4) with corresponding densities $w_\epsilon(x,t) = \int_V g_\epsilon(x,v,t)\, dv$ and $W_\epsilon(x,t) = \int_V G_\epsilon(x,v,t)\, dv$; and let $g_\epsilon^0, g_{\epsilon 0}, g_{\epsilon 1}$ resp. $G_\epsilon^0, G_{\epsilon 0}, G_{\epsilon 1}$ denote the corresponding data. We have

$$\begin{aligned} &\|g_\epsilon - G_\epsilon\|_{L^1(\Omega\times V)} + \|a(v)\cdot n(g_\epsilon - G_\epsilon)\|_{L^1(\Gamma_0^+ \times (0,t))} + \|a(v)\cdot n(g_\epsilon - G_\epsilon)\|_{L^1(\Gamma_1^+\times(0,t))} \\ &\quad \leqslant \|g_\epsilon^0 - G_\epsilon^0\|_{L^1(\Omega\times V)} + \|a(v)\cdot n(g_{\epsilon 0} - G_{\epsilon 0})\|_{L^1(\Gamma_0^-\times(0,t))} \\ &\quad\quad + \|a(v)\cdot n(g_{\epsilon 1} - G_{\epsilon 1})\|_{L^1(\Gamma_1^-\times(0,t))} \end{aligned} \tag{10}$$

**Remark 2.1.** Although we can derive contraction properties directly from the integral representation, we prefer to use a different method, which allows us to obtain the inequalities in (10).

*Proof of Theorem 2.1.* We begin with proving the uniqueness and the continuous dependence of the solution on the data given in (10). These estimates are needed for the proofs of various results in this paper. Therefore, we shall give a somewhat detailed proof. The idea of the proof is to use a combination of the author's method[10, 11] and ideas from ref. 4.

The function $G_\epsilon$ satisfies an equation similar to Eq. (1). Subtracting this equation from Eq. (1), and multiplying the resulting equation by $\varphi$ a test function in $C^1(\bar{\Omega}\times V\times[0,T])$ to be precised later, and integrating by parts, we obtain

$$\begin{aligned}
&\int_{\Omega\times V}((g_\epsilon-G_\epsilon)\,\varphi)(\cdot,\cdot,t)-\int_{\Omega\times V}((g_\epsilon-G_\epsilon)\,\varphi)(\cdot,\cdot,0)\\
&\quad-\int_{\Omega\times V\times(0,t)}(\partial_t+a(v)\cdot\partial_x)(\varphi)(g_\epsilon-G_\epsilon)\\
&\quad+\int_{\Gamma_0^-\times(0,t)}a(v)\cdot n(g_{\epsilon 0}-g_{\epsilon 0})\,\varphi+\int_{\Gamma_1^-\times(0,t)}a(v)\cdot n(g_{\epsilon 1}-G_{\epsilon 1})\,\varphi\\
&\quad+\int_{\Gamma_0^+\times(0,t)}a(v)\cdot n(g_\epsilon-G_\epsilon)\,\varphi+\int_{\Gamma_1^+\times(0,t)}a(v)\cdot n(g_\epsilon-G_\epsilon)\,\varphi\\
&=\frac{1}{\epsilon}\int_{\Omega\times V\times(0,t)}((\chi_{w_\epsilon}-\chi_{W_\epsilon})-(g_\epsilon-G_\epsilon))\,\varphi
\end{aligned}\tag{11}$$

We then take $\varphi=\operatorname{sign}^\mu(g_\epsilon-G_\epsilon)\,\psi(x,t)$ with $x\operatorname{sign}^\mu(x)\geqslant 0\quad x\in\mathbb{R}$, and $\psi$ is a nonnegative test function and $\operatorname{sign}^\mu$ is a regularization of sign function. Plugging in (11) and passing to the limit as $\mu\to 0$, we obtain

$$\begin{aligned}
&\int_{\Omega\times V}(|g_\epsilon-G_\epsilon|\,\psi)(\cdot,\cdot,t)+\int_{\Gamma_0^+\times(0,t)}a(v)\cdot n\,|g_\epsilon-G_\epsilon|\,\psi\\
&\quad+\int_{\Gamma_1^+\times(0,t)}a(v)\cdot n\,|g_\epsilon-G_\epsilon|\,\psi+\int_{\Gamma_0^-\times(0,t)}a(v)\cdot n\,|g_{\epsilon 0}-G_{\epsilon 0}|\,\psi\\
&\quad+\int_{\Gamma_1^-\times(0,t)}a(v)\cdot n\,|g_{\epsilon 1}-G_{\epsilon 1}|\,\psi-\int_{\Omega\times V}(|g_\epsilon-G_\epsilon|\,\psi)(\cdot,\cdot,0)\\
&=\frac{1}{\epsilon}\int_{\Omega\times V\times(0,t)}[(\chi_{w_\epsilon}-\chi_{W_\epsilon})\operatorname{sign}(g_\epsilon-G_\epsilon)-|g_\epsilon-G_\epsilon|]\,\psi\\
&\quad+\int_{\Omega\times V\times(0,t)}(\partial_t+a(v)\cdot\partial_x)(\psi)\,|g_\epsilon-G_\epsilon|
\end{aligned}\tag{12}$$

Using the properties of $\chi$, this yields

$$\int_{\Omega\times V} (|g_\epsilon - G_\epsilon|\,\psi)(\cdot,\cdot,t) + \int_{\Gamma_0^+\times(0,t)} a(v)\cdot n\,|g_\epsilon - G_\epsilon|\,\psi$$
$$+ \int_{\Gamma_1^+\times(0,t)} a(v)\cdot n\,|g_\epsilon - G_\epsilon|\,\psi$$
$$\leqslant \int_{\Omega\times V} (|g_\epsilon - G_\epsilon|\,\psi)(\cdot,\cdot,0) - \int_{\Gamma_0^-\times(0,t)} a(v)\cdot n\,|g_{\epsilon 0} - G_{\epsilon 0}|\,\psi$$
$$- \int_{\Gamma_1^-\times(0,t)} a(v)\cdot n\,|g_{\epsilon 1} - G_{\epsilon 1}|\,\psi + \int_{\Omega\times V\times(0,t)} (\partial_t + a(v)\cdot\partial_x)(\psi)\,|g_\epsilon - G_\epsilon| \tag{13}$$

Taking now $\psi(t)\equiv 1$ yields the estimate (10).

To prove the existence of a solution to the kinetic problem, we use the following iterations

$$[\partial_t + a(v)\cdot\partial_x]\, g_\epsilon^{n+1}(x,v,t)$$
$$= \frac{1}{\epsilon}\,(\chi_{w_\epsilon^n(x,t)}(v) - g_\epsilon^{n+1}(x,v,t)) \qquad \text{in} \quad \Omega\times V\times(0,T) \tag{14}$$

$$g_\epsilon^{n+1}(x,v,t) = g_{\epsilon 0}(x,v,t) \qquad \text{on} \quad \Gamma_0^-\times(0,T) \tag{15}$$

$$g_\epsilon^{n+1}(x,v,t) = g_{\epsilon 1}(x,v,t) \qquad \text{on} \quad \Gamma_1^-\times(0,T), \tag{16}$$

$$g_\epsilon^{n+1}(x,v,0) = g_\epsilon^0(x,v) \qquad \text{in} \quad \Omega\times V \tag{17}$$

Using (12) in the present context with $g_\epsilon = g_\epsilon^{n+1}$ and $G_\epsilon = g_\epsilon^{m+1}$, and using the properties of $\chi$, we obtain

$$\int_{\Omega\times V} (|g_\epsilon - G_\epsilon|\,\psi)(\cdot,\cdot,t) + \int_{\Gamma_0^+\times(0,t)} a(v)\cdot n\,|g_\epsilon - G_\epsilon|\,\psi$$
$$+ \int_{\Gamma_1^+\times(0,t)} a(v)\cdot n\,|g_\epsilon - G_\epsilon|\,\psi + \frac{1}{\epsilon}\int_{\Omega\times V\times(0,t)} |g_\epsilon - G_\epsilon|\,\psi$$
$$= \frac{1}{\epsilon}\int_{\Omega\times V\times(0,t)} (\chi_{w_\epsilon^n} - \chi_{W_\epsilon^m})\,\mathrm{sign}(g_\epsilon - G_\epsilon)\,\psi$$
$$+ \int_{\Omega\times V\times(0,t)} (\partial_t + a(v)\cdot\partial_x)(\psi)\,|g_\epsilon - G_\epsilon|$$
$$\leqslant \int_{\Omega\times V\times(0,t)} (\partial_t + a(v)\cdot\partial_x)(\psi)\,|g_\epsilon - G_\epsilon| + \frac{1}{\epsilon}\int_{\Omega\times V\times(0,t)} |g_\epsilon^n - g_\epsilon^m|\,\psi \tag{18}$$

Taking $\psi = e^{-\frac{\alpha}{\epsilon}s}$, $0 \leqslant s \leqslant t$, with $\alpha$ a positive constant, we then obtain

$$\int_{\Omega\times V} (|g_\epsilon^{n+1} - g_\epsilon^{m+1}|\,\psi)(\cdot,\cdot,t) + \int_{\Gamma_0^+\times(0,t)} a(v)\cdot n\,|g_\epsilon^{n+1} - g_\epsilon^{m+1}|\,\psi$$

$$+\int_{\Gamma_1^+\times(0,t)} a(v)\cdot n\,|g_\epsilon^{n+1} - g_\epsilon^{m+1}|\,\psi + \frac{1+\alpha}{\epsilon}\int_{\Omega\times V\times(0,t)} |g_\epsilon^{n+1} - g_\epsilon^{m+1}|\,\psi$$

$$\leqslant \frac{1}{\epsilon}\int_{\Omega\times V\times(0,t)} |g_\epsilon^{n} - g_\epsilon^{m}|\,\psi \tag{19}$$

Hence we obtain

$$\int_{\Omega\times V\times(0,t)} \psi\,|g_\epsilon^{n+1} - g_\epsilon^{m+1}| \leqslant \frac{1}{1+\alpha}\int_{\Omega\times V\times(0,t)} |g_\epsilon^{n} - g_\epsilon^{m}|\,\psi \tag{20}$$

This and a reuse of (19) proves that the iterations are contracted to the unique fixed point in $L^\infty([0,T]; L^1(\Omega\times V))$, which satisfies Eq. (1) and also the boundary and initial conditions (2)–(4). We also infer from the inequality (10) that the solution $g_\epsilon$ depends continuously on the initial and boundary data.

The integral representation is obtained using the characteristic method. The proof of the theorem is now finished.

## 2.2. Kinetic Entropy

We shall prove an entropy inequality for the solution of the kinetic problem. This is stated in the following theorem.

**Theorem 2.2.** The solution to the kinetic problem satisfies the relation

$$-\int_{\Omega\times V\times(0,T)} (\partial_t + a(v)\cdot\partial_x)(\psi)\,|g_\epsilon - \chi_k| + \int_{\Gamma_0^-\times(0,T)} a(v)\cdot n\psi\,|g_{\epsilon 0} - \chi_k|$$

$$+\int_{\Gamma_1^-\times(0,T)} a(v)\cdot n\psi\,|g_{\epsilon 1} - \chi_k| \leqslant 0$$

$$\forall \psi \in C_0^1(\bar\Omega\times V\times(0,T)),\quad \psi \geqslant 0,\quad \forall k\in\mathbb{R} \tag{21}$$

*Proof.* Multiplying Eq. (1) by $\varphi = \mathrm{sign}^\mu(g_\epsilon - \chi_k)\,\psi(x,t)$ with $\mathrm{sign}^\mu(x)$ the regularization of sign function mentioned in the proof of Theorem 2.1, and $\psi$ is a nonnegative test function in $C_0^1(\bar\Omega\times V\times(0,T))$, and proceeding as in the proof of Theorem 2.1, and using the properties of $\chi_w$, we obtain the entropy inequality (21).

## 2.3. Basic Estimates of the Solution

We shall state and prove here some basic estimates for the solution of the kinetic problem. These are essential for the study of the hydrodynamic limit. We begin with $L^\infty$ estimates.

**Lemma 2.1.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega \times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3$$

with $C_1$, $C_2$, and $C_3$ positive constants independent of $\epsilon$. Then $g_\epsilon$ is uniformly bounded in $L^\infty(\Omega \times V \times [0,T])$. Moreover we have

$$\|g_\epsilon\|_\infty \leqslant \max(\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])}, \|g_\epsilon^0\|_{L^\infty(\Omega \times V)}, \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])}) + 1$$

*Proof.* The proof is based on the use of the integral representation of the solution respectively on $\Omega_0$, $\Omega_{01}$, and $\Omega_1$.

We now present estimates of $g_\epsilon$ and $w_\epsilon$ in $L^\infty([0,T]; L^1(\Omega \times V))$ and $L^\infty([0,T]; L^1(\Omega))$ respectively.

**Lemma 2.2.** Assume that

$$\|a(v) \cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^- \times (0,T))} < C_1, \qquad \|g_\epsilon^0\|_{L^1(\Omega \times V)} < C_2,$$
$$\|a(v) \cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^- \times (0,T))} < C_3$$

with $C_1$, $C_2$, and $C_3$ positive constants independent of $\epsilon$. Then $g_\epsilon$ is uniformly bounded in $L^\infty([0,T]; L^1(\Omega \times V))$ and $w_\epsilon$ is uniformly bounded in $L^\infty([0,T]; L^1(\Omega))$. Moreover, we have

$$\begin{aligned}\|w_\epsilon\|_{L^\infty([0,T]; L^1(\Omega))} &\leqslant \|g_\epsilon\|_{L^\infty([0,T]; L^1(\Omega \times V))} \\ &\leqslant \|a(v) \cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^- \times (0,T))} + \|a(v) \cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^- \times (0,T))} \\ &\quad + \|g_\epsilon^0\|_{L^1(\Omega \times V)}\end{aligned}$$

*Proof.* Using Formula (10) with $G_\epsilon \equiv 0$, we obtain

$$\int_{\Omega \times V} |g_\epsilon(x, v, t)| \leqslant \int_{\Omega \times V} |g_\epsilon^0(x, v)| + \int_{\Gamma_0^- \times (0,T)} |a(v) \cdot n g_{\epsilon 0}|$$
$$+ \int_{\Gamma_1^- \times (0,T)} |a(v) \cdot n g_{\epsilon 1}|$$

The lemma then follows.

Next we shall show that under the conditions that the supports in $v \in V$ of the data are compact, the supports in $v \in V$ of $g_\epsilon$ remain compactly supported with supports included in a fixed compact set independent of $\epsilon$. We shall also give some information about the speed of propagation $a(v)$. This is stated in the following lemma.

**Lemma 2.3.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega \times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3$$

with $C_1$, $C_2$, and $C_3$ positive constants independent of $\epsilon$. Assume also that the initial and boundary data $g_\epsilon^0$, $g_{\epsilon 0}$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$. Then

(i) $w_\epsilon$ is uniformly bounded in $L^\infty(\Omega \times [0,T])$.

(ii) $g_\epsilon$ remains compactly supported in $v \in V$ with support included in a fixed compact set independent of $\epsilon$.

(iii) The speed of propagation $a(v)$ is finite.

**Remark 2.2.** In ref. 7 the uniform $L^\infty$ boundedness (in $\epsilon$) of the macroscopic density $u_\epsilon = \int_V f_\epsilon(x, v, t)\, dv$ and hence the compactness of the support in $v$ of $f_\epsilon(t, x, v)$ together with the finite speed of propagation remained unproven. Since in their proof, which is given on p. 504 lines 6 through 12 of ref. 7, their argument is incorrect. Following we quote lines 6 through 12 of p. 504 of ref. 7.

"2. Finite speed of propagation. We assume that initially, $f_\epsilon(x, \cdot, 0)$ has a compact support in $\mathbb{R}_v$. Let us first show that $f_\epsilon(x, \cdot, t)$ remains compactly supported. Indeed, by (2.6), $f_\epsilon(x, v, t)$ and hence $u_\epsilon(\cdot, t)$ are uniformly bounded, and therefore the contributions of $\chi_{u(\cdot,\cdot)}(v)$ on the right hand side of (2.2) are supported by $v \in [-u_\infty, u_\infty]$, where $u_\infty = \|u_\epsilon(x,t)\|_{L^\infty(\mathbb{R}_x^d \times \mathbb{R}_t^+)}$. Consequently, $f_\epsilon(x, \cdot, t)$ given in 2.2 remains compactly supported for all $t > 0$, with support contained in $\mathrm{supp}_v\, f_\epsilon(x, \cdot, 0) \cup [-u_\infty, u_\infty] \cdots$."

The argument: Indeed, by (2.6), $f_\epsilon(x, v, t)$ and hence $u_\epsilon(\cdot, t)$ are uniformly bounded, is incorrect since the uniform (in $\epsilon$) boundedness of a function (here $f_\epsilon(x, v, t)$) in $L^\infty(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+)$ does not in general yield the uniform boundedness (in $\epsilon$) of its velocity average (here $u_\epsilon(x, t) = \int_{\mathbb{R}} f_\epsilon(x, v, t)\, dv$). Take for example the function $h_\epsilon(x, v, t) = e^{-\epsilon |v|} \times \exp(-t - \sum |x_i|)$ and its velocity average $u_\epsilon(x, t) = \frac{2}{\epsilon} \exp(-t - \sum |x_i|)$.

Since the main results of the paper of ref. 7 use in a fundamental way the finite speed of propagation together with the compactness of the support in the velocity $v$ of the microscopic density $f_\epsilon(t, x, v)$, which are in turn based on the uniform $L^\infty$ boundedness (in $\epsilon$) of the macroscopic density $u_\epsilon = \int_V f_\epsilon(x, v, t)\, dv$, the paper of ref. 7 is not correct.

In ref. 8, in order to obtain the uniform (in $\epsilon$) bound of $u_\epsilon(x, t) = \int_{\mathbb{R}} f_\epsilon(x, v, t)\, dv$ in $L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$, and hence to fill the gap of ref. 7, the author assumed an additional assumption on the sign of the data: $f_\epsilon(\cdot, v, 0)\, \mathrm{sign}(v) \geqslant 0$. It is clear that this assumption is very restrictive.

Because of the above it is clear that the general proof of the above results remained open despite the various attempts by various authors. We shall give below two different proofs. One is general and does not use any additional assumptions, thus solves also the gap in ref. 7, and the second relies on the additional assumption on the sign of the data, and thus allows us to compare the two proofs. The first proof of (i) given below solves the gap in ref. 7 and allows us, fortunately, to conclude that the results in ref. 7 are true although their proofs given in ref. 7 are not correct.

*Proof.* (i) *First and general proof of the uniform in $\epsilon$ $L^\infty$ bound.*

Before we give the proof we shall outline the main steps. Let

$$A_\epsilon = \{(x, v, t) \in \Omega \times V \times (0, T) \mid |w_\epsilon(x, t)| > |v|\}$$

$$V_\epsilon = \{v \in V \mid (x, v, t) \in A_\epsilon \text{ for some } (x, t) \in \Omega \times (0, T)\}$$

Let $|F|$ denote the Lebesgue measure of the set $F$. Let $m_\epsilon$ and $n_\epsilon$ denote the Lebesgue measure of $A_\epsilon$ respectively $V_\epsilon$. Let $C_0 > 0$ be a fixed positive constant. Let $\Upsilon$ denote the set of all $\epsilon > 0$ such that

$$\|w_\epsilon\|_\infty > C_0$$

We prove in a first step that there exists $\gamma > 0$ such that $\gamma < m_\epsilon < C_1$ uniformly in $\epsilon \in \Upsilon$. We then deduce that $n_\epsilon = |V_\epsilon| < C_2$ uniformly in $\epsilon \in \Upsilon$ where $C_2$ is a positive constant. In the third step we show that this uniform boundedness implies that

$$\int_V \left( \int_{\Omega \times (0,T)} |g_\epsilon(x, v, t)|^p\, dx\, dt \right)^{1/p} dv < C_3$$

uniformly in $\epsilon \in \Upsilon$. Here $p$ is arbitrarily large. Finally, using Minkowski inequality, we obtain

$$\left( \int_{\Omega \times (0,T)} \left( \int_V |g_\epsilon(x, v, t)|\, dv \right)^p dx\, dt \right)^{1/p} \leqslant \int_V \left( \int_{\Omega \times (0,T)} |g_\epsilon(x, v, t)|^p\, dx\, dt \right)^{1/p}$$

$$\leqslant C_3$$

Thus, we obtain $\|\int_V |g_\epsilon|\, dv\|_{L^p(\Omega\times(0,T))} \leqslant C_3$ uniformly in $\epsilon \in \Upsilon$. By the properties of the $L^\infty$ norm, as $p \to \infty$, we conclude that $\|\int_V |g_\epsilon|\, dv\|_{L^\infty(\Omega\times(0,T))} \leqslant C_3$ uniformly in $\epsilon \in \Upsilon$. Therefore, $\|\int_V |g_\epsilon|\, dv\|_{L^\infty(\Omega\times(0,T))}$ is uniformly bounded in $\epsilon$ and hence $w_\epsilon$ is uniformly in $\epsilon$ bounded in $L^\infty(\Omega \times (0,T))$.

*Step 1.* We first notice that for every fixed $\epsilon$, using Gronwall lemma we conclude that $g_\epsilon$ is in $L^\infty(\Omega\times(0,T); L^1(V))$ and hence $w_\epsilon$ is in $L^\infty(\Omega\times(0,T))$. Observe that such argument does not provide a uniform in $\epsilon$ bound of $g_\epsilon$ in $L^\infty(\Omega\times(0,T); L^1(V))$.

Let $m_\epsilon$ and $n_\epsilon$ denote the Lebesgue measure of $A_\epsilon$ respectively $V_\epsilon$. We know from Lemma 2.2 that

$$m_\epsilon = \int_{\Omega\times V\times(0,T)} |\chi_{w_\epsilon(x,t)}(v)|\, dx\, dv\, dt = \int_{\Omega\times(0,T)} |w_\epsilon(x,t)|\, dx\, dt < C \tag{22}$$

where $C$ is independent of $\epsilon$.

Let $C_0 > 0$ be a fixed constant. Let $\Upsilon$ denote the set of all $\epsilon > 0$ such that

$$\|w_\epsilon\|_\infty > C_0 \tag{23}$$

We know from the begining of this proof that $w_\epsilon$ is in $L^\infty(\Omega\times(0,T))$ for every fixed $\epsilon$. If the set $\Upsilon$ is empty or finite then the proof will be concluded easily. Therefore, we assume that $\Upsilon$ is neither empty nor finite.

In this step we prove the following statement

$$\exists\gamma > 0 \text{ such that } \gamma < m_\epsilon \text{ uniformly in } \epsilon \in \Upsilon \tag{24}$$

We begin by proving the following statement.

$$\begin{gathered}\exists\beta \text{ with } 0<\beta<C_0,\ \exists E\subset\Omega\times(0,T) \text{ with } |E|>0\\ \text{such that } \|w_\epsilon\|_{\infty,E} > \beta \text{ uniformly in } \epsilon\in\Upsilon\end{gathered} \tag{25}$$

If (25) is not true then

$$\begin{gathered}\forall\beta \text{ with } 0<\beta<C_0,\ \forall E\subset\Omega\times(0,T) \text{ with } |E|>0,\\ \exists\epsilon\in\Upsilon \text{ such that } \|w_\epsilon\|_{\infty,E} \leqslant \beta\end{gathered} \tag{26}$$

Thus, taking $\beta = C_0 - \frac{1}{n}$ and $E = \Omega\times(0,T)$, there exists $\epsilon_n$ a subsequence in $\Upsilon$ such that $|w_{\epsilon_n}(y)| \leqslant C_0 - \frac{1}{n}$, a.e. $y \in E$. This implies that $\|w_{\epsilon_n}\|_\infty \leqslant C_0$ with $\epsilon_n \in \Upsilon$. This contradicts (23). Therefore, (25) is true.

If the set $E$ in (25) is of infinite measure, then we can select a subset $E'$ of $E$ with $0 < |E'| < \infty$ such that (25) holds for $E'$. If not then

$$\forall \beta \text{ with } 0 < \beta < C_0, \ \forall E' \subset \Omega \times (0, T) \text{ with } 0 < |E'| < \infty,$$
$$\exists \epsilon \in \Upsilon \text{ such that } \|w_\epsilon\|_{\infty, E'} \leqslant \beta \tag{27}$$

Thus there is a subsequence $w_{\epsilon_n}$ satisfying $\|w_{\epsilon_n}\|_{\infty, E_n} \to 0$ as $n \to \infty$ with $E_n$ an increasing sequence of subsets $E_n$ of $E$ ($E_n \subset E_{n+1} \subset E$) of finite measures such that $\bigcup_n E_n = E$. Hence for any $\delta > 0$ there is an integer $N_\delta$ such that for $n \geqslant N_\delta$ we have $\|w_{\epsilon_n}\|_{\infty, E_n} < \delta$. Now let $m > N_\delta$, since in particular $\|w_{\epsilon_n, E}\|_\infty > \beta$ uniformly in $n$ (consult (25)), $\|w_{\epsilon_m, E}\|_\infty > \beta$. By the properties of the $L^\infty$ norm there is a bounded subset $F_m$ of $E$ with small positive measure such that $|w_{\epsilon_m}(x)| > \beta$ for a.e. $x \in F_m$. On the other hand there is an integer $p \geqslant N_\delta$ such that $F_m \subset E_p$. Therefore, we have $\int_{F_m} |w_{\epsilon_m}| > |F_m| \, \beta$ and $\int_{F_m} |w_{\epsilon_m}| \leqslant |F_m| \, \|w_{\epsilon_m}\|_{\infty, E_p} < \delta \, |F_m|$. We then obtain $\beta < \delta$. Since $\delta$ was arbitrary we obtain a contradiction. So we may assume that the set $E$ in (25) satisfies $0 < |E| < \infty$. This is important since we will use below Egoroff theorem for sequence defined on such set $E$.

We now prove that (24) is true. Assume to the contrary that (24) is not true. Then there is a subsequence $\epsilon_k$ in $\Upsilon$ such that $m_{\epsilon_k} \xrightarrow[k \to \infty]{} 0$. But we have

$$m_{\epsilon_k} = \int_{\Omega \times V \times (0, T)} |\chi_{w_{\epsilon_k}(x, t)}(v)| \, dx \, dv \, dt = \int_{\Omega \times (0, T)} |w_{\epsilon_k}(x, t)| \, dx \, dt$$

Hence $\int_{\Omega \times (0, T)} |w_{\epsilon_k}(x, t)| \, dx \, dt \to 0$. Therefore there is a subsequence $w_{\epsilon_{k_n}}$ that converges a.e. to 0 on $\Omega \times (0, T)$. In particular, $w_{\epsilon_{k_n}} \to 0$ on $E$, where $E$ is the set given in (25). Using Egoroff theorem,[3] $w_{\epsilon_{k_n}} \to 0$ almost uniformly on $E$ (recall from above that $E$ can be selected to satisfy $0 < |E| < \infty$). That is, $\forall \eta > 0$, $\exists E_\eta \subset E$ such that $|E \setminus E_\eta| < \eta$ and $w_{\epsilon_{k_n}} \to 0$ uniformly on $E_\eta$. Now fix $\eta > 0$ small and let $\delta > 0$ be given, then there is $n'$ depending on $\delta$ such that

$$|w_{\epsilon_{k_n}}(y)| < \delta \qquad \forall y \in E_\eta, \quad \forall \epsilon_{k_n} < \epsilon_{k_{n'}} \tag{28}$$

Now let

$$\tilde{E} = \{x \in E : |w_{\epsilon_{k_n}}(x)| > \beta \ \forall \epsilon_{k_n} < \epsilon_{k_{n'}}\} \tag{29}$$

then (25) implies that $|\tilde{E}| > \alpha > 0$ for some $\alpha > 0$. Now choose $\eta < \alpha$ then $E_\eta$ must contain a subset $\hat{E} \subset \tilde{E}$ with $|\hat{E}| > 0$. For otherwise the set $F = \tilde{E} \setminus \tilde{\tilde{E}}$ where

$$\tilde{\tilde{E}} = \{x \in \tilde{E} \cap E_\eta : |w_{\epsilon_{k_n}}(x)| > \beta \ \forall \epsilon_{k_n} < \epsilon_{k_{n'}}\}, \qquad \text{and} \quad |\tilde{\tilde{E}}| = 0$$

is included in $E \setminus E_\eta$ ($F \subset E \setminus E_\eta$) and $\alpha < |F| \leqslant |E \setminus E_\eta| < \eta < \alpha$ which is impossible. Now pick $\delta < \beta$ in (28). Then in particular, we obtain

$$\|w_{\epsilon_{k_n}}\|_{\infty, \hat{E}} < \beta \qquad \forall \epsilon_{k_n} < \epsilon_{k_{n'}}$$

which is a contradiction to (29). Therefore, (24) is true.

*Step 2.* We prove that $|V_\epsilon| \leqslant C$ uniformly in $\epsilon \in \Upsilon$.

Step 1 yields $0 < \gamma < m_\epsilon = |A_\epsilon| < C\ \forall \epsilon \in \Upsilon$ (consult (22) and (24)). Now using the regularity of the Lebesgue measure, we have for any $\eta > 0$ such that $\gamma - \eta > 0$, there exist a compact set $F_\epsilon^\eta$ and an open set $U_\epsilon^\eta$ such that $F_\epsilon^\eta \subset A_\epsilon \subset U_\epsilon^\eta$ and $|A_\epsilon| - \eta < |F_\epsilon^\eta| < |A_\epsilon|$ and $|A_\epsilon| < |U_\epsilon^\eta| < |A_\epsilon| + \eta < C + \eta$. Thus for $\eta < \gamma/2$, there are $F_\epsilon^\eta$ and $U_\epsilon^\eta$ so that

$$0 < \gamma/2 < |F_\epsilon^\eta| \leqslant |A_\epsilon| \leqslant |U_\epsilon^\eta| < C + \gamma/2 \qquad \forall \epsilon \in \Upsilon \tag{30}$$

Above we have used (22) and (24). Now by Vitali's Covering Theorem,[3] there exists a countable collection $\mathscr{G}_\epsilon$ of disjoint closed balls in $U_\epsilon^\eta$ such that diam $B \leqslant \eta < \frac{\gamma}{2}$ for all $B \in \mathscr{G}_\epsilon$ and $|U_\epsilon^\eta - \bigcup_{B \in \mathscr{G}_\epsilon} B| = 0$. Using (30) above, we then conclude that $|\bigcup_{B \in \mathscr{G}_\epsilon} B|$ is bounded below and above by positive constants independent of $\epsilon \in \Upsilon$. Thus the projection $V_\epsilon$ of $A_\epsilon$ with respect to the $v$ axis has a one dimensional Lebesgue measure which is bounded above by a positive constant independent of $\epsilon \in \Upsilon$. This proves the fact that $n_\epsilon = |V_\epsilon| < C$ with $C$ a constant independent of $\epsilon \in \Upsilon$.

*Step 3.* We prove that

$$\int_V \left( \int_{\Omega \times (0,T)} |g_\epsilon(x, v, t)|^p \, dx \, dt \right)^{1/p} dv < C$$

where $C$ is a positive constant independent of $\epsilon$.

We write the integral representation of the solution $g_\epsilon$ in $\Omega_0$ in the form

$$g_\epsilon(x, v, t) = g_{\epsilon 0}\left( x_\star - \frac{x_1}{a_1(v)}\, a_\star(v), v, t - \frac{x_1}{a_1(v)} \right) \exp(-x_1/(a_1(v)\,\epsilon))$$
$$+ (1 - \exp(-x_1/(a_1(v)\,\epsilon)))\, \frac{\int_{t-\frac{x_1}{a_1(v)}}^{t} e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s), s)}(v)\, ds}{\int_{t-\frac{x_1}{a_1(v)}}^{t} e^{(s-t)/\epsilon}\, ds}$$

Thus $g_\epsilon(x, v, t)$ is expressed as a convex combination. So by Jensen inequality, we obtain for any convex function $\varphi(g_\epsilon)$,

$$\varphi(g_\epsilon(x,v,t)) \leqslant \varphi\left(g_{\epsilon 0}\left(x_\star - \frac{x_1}{a_1(v)}\, a_\star(v), v, t - \frac{x_1}{a_1(v)}\right)\right) \exp(-x_1/(a_1(v)\,\epsilon))$$
$$+ \frac{1}{\epsilon}\int_{t-\frac{x_1}{a_1(v)}}^{t} e^{(s-t)/\epsilon} \varphi(\chi_{w_\epsilon(x(s),\,s)}(v))\, ds$$

We obtain similar formula for $g_\epsilon(x, v, t)$ in $\Omega_{01}$ and $\Omega_1$. Now taking $\varphi(g) = |g|^p$ and integrating over $x$ and $t$, we obtain

$$\int_{\Omega\times(0,T)} |g_\epsilon(x,v,t)|^p\, dx\, dt$$
$$\leqslant \int_{\Gamma_0\times(0,T)} |g_{\epsilon 0}(y,v,t)|^p\, dy\, dt + \int_\Omega |g_\epsilon^0(x,v))|^p\, dx$$
$$+ \int_{\Gamma_1\times(0,T)} |g_{\epsilon 1}(y,v,t)|^p\, dy\, dt + \int_{\Omega\times(0,T)} \frac{1}{\epsilon}\int_0^t e^{(s-t)/\epsilon}\, |\chi_{w_\epsilon(x,\,s)}|\, dx\, ds\, dt$$

Taking the $p$-root of both sides and integrating over $V$, we obtain

$$\int_V \left(\int_{\Omega\times(0,T)} |g_\epsilon(x,v,t)|^p\, dx\, dt\right)^{1/p} dv$$
$$\leqslant 4^{1/p} \max\left[\int_V \left(\int_{\Gamma_0\times(0,T)} |g_{\epsilon 0}(x,v,t)|^p\right)^{1/p} dv, \int_V \left(\int_\Omega |g_\epsilon^0(x,v))|^p\, dx\right)^{1/p} dv,\right.$$
$$\int_V \left(\int_{\Gamma_1\times(0,T)} |g_{\epsilon 1}(x,v,t)|^p\, dx\, dt\right)^{1/p} dv,$$
$$\left.\int_V \left(\int_{\Omega\times(0,T)} \frac{1}{\epsilon}\int_0^t e^{(s-t)/\epsilon}\, |\chi_{w_\epsilon(x,\,s)}|\, dx\, ds\, dt\right)^{1/p} dv\right] \tag{31}$$

We only need to prove that

$$\int_V \left(\int_{\Omega\times(0,T)} \frac{1}{\epsilon}\int_0^t e^{(s-t)/\epsilon}\, |\chi_{w_\epsilon(x,\,s)}|\, dx\, ds\, dt\right)^{1/p} dv$$

is bounded uniformly in $\epsilon$ for $p$ arbitrarily large. The other terms in (31) are clearly bounded uniformly in $\epsilon$ for $p$ arbitrarily large. For example, the term $\int_V (\int_{\Gamma_0\times(0,T)} |g_{\epsilon 0}(x,v,t)|^p)^{1/p}\, dv$ is uniformly bounded for $p$ arbitrarily large since by assumption $g_{\epsilon 0}$ is uniformly bounded in $\epsilon$ in $L^\infty(\Gamma_0\times(0,T)\times L^1(V))$ and similarly for $\int_V (\int_\Omega |g_\epsilon^0(x,v)|^p)^{1/p}\, dv$ and $\int_V (\int_{\Gamma_1\times(0,T)} |g_{\epsilon 1}(x,v,t)|^p\, dx\, dt)^{1/p}\, dv$.

Now we have

$$\int_{\Omega\times(0,T)} \frac{1}{\epsilon}\int_0^t e^{(s-t)/\epsilon}\,|\chi_{w_\epsilon(x,s)}|\,ds\,dt\,dx = \int_{\Omega\times(0,T)} |\chi_{w_\epsilon(x,s)}|\,(1-e^{(s-T)/\epsilon})\,ds\,dx$$

Thus, we have

$$\begin{aligned}
&\int_V \left(\int_{\Omega\times(0,T)} \frac{1}{\epsilon}\int_0^t e^{(s-t)/\epsilon}\,|\chi_{w_\epsilon(x,s)}|\,dx\,ds\,dt\right)^{1/p} dv \\
&\quad = \int_V \left(\int_{\Omega\times(0,T)} |\chi_{w_\epsilon(x,s)}|\,(1-e^{(s-T)/\epsilon})\,ds\,dx\right)^{1/p} dv \\
&\quad = \int_{V_\epsilon} \left(\int_{\Omega\times(0,T)} |\chi_{w_\epsilon(x,s)}|\,(1-e^{(s-T)/\epsilon})\,ds\,dx\right)^{1/p} dv \\
&\quad \leqslant \left(\int_{V_\epsilon}\int_{\Omega\times(0,T)} |\chi_{w_\epsilon(x,s)}|\,(1-e^{(s-T)/\epsilon})\,ds\,dx\,dv\right)^{1/p} (n_\epsilon)^{1/p'} \\
&\quad \leqslant C^{1/p}C^{1/p'} = C
\end{aligned} \tag{32}$$

with $C$ independent of $\epsilon$. Above we have used Lemma 2.2, Holder inequality, and the uniform boundedness of $n_\epsilon = |V_\epsilon|$ obtained in Step 2.

Using this in (31), we conclude that

$$\begin{aligned}
&\int_V \left(\int_{\Omega\times(0,T)} |g_\epsilon(x,v,t)|^p\,dx\,dt\right)^{1/p} dv \\
&\quad \leqslant 4^{1/p}\max\Bigg[\int_V \left(\int_{\Gamma_0\times(0,T)} |g_{\epsilon 0}(x,v,t)|^p\right)^{1/p} dv, \\
&\qquad\qquad \int_V \left(\int_\Omega |g_\epsilon^0(x,v))|^p\,dx\right)^{1/p} dv, \\
&\qquad\qquad \int_V \left(\int_{\Gamma_1\times(0,T)} |g_{\epsilon 1}(x,v,t)|^p\,dx\,dv\right)^{1/p} dv,\ C^{1/p}C^{1/p'}\Bigg] \\
&\quad \leqslant C
\end{aligned} \tag{33}$$

*Step 4.* We conclude here that $g_\epsilon$ is uniformly in $\epsilon$ bounded in $L^\infty(\Omega\times(0,T);L^1(V))$. Therefore its velocity average $w_\epsilon$ is uniformly in $\epsilon$ bounded in $L^\infty(\Omega\times(0,T))$.

Using Minkowski inequality,[5] we have

$$\left(\int_{\Omega\times(0,T)}\left(\int_V |g_\epsilon|\,dv\right)^p dx\,dt\right)^{1/p} \leqslant \int_V\left(\int_{\Omega\times(0,T)} |g_\epsilon(x,v,t)|^p\,dx\,dt\right)^{1/p} dv \tag{34}$$

Taking the limit as $p \to \infty$ in (33) and (34), we conclude that $\|\int_V |g_\epsilon| \, dv\|_{L^\infty(\Omega \times (0,T))}$ is uniformly in $\epsilon$ bounded and hence $w_\epsilon$ is also uniformly in $\epsilon$ bounded in $L^\infty(\Omega \times (0,T))$. This concludes the proof of (i).

**Remark 2.3.** In fact we could have proved this result by reasoning by contradiction. At the begining of Step 1, we assume that the family $\|w_\epsilon\|_\infty$ is not uniformly in $\epsilon$ bounded, then there is a subsequence $w_{\epsilon_n}$ such that $\|w_{\epsilon_n}\|_\infty \to \infty$ as $n \to \infty$. So for any $A > 0$ there is an integer $N_A$ such that for all $n \geqslant N_A$, $\|w_{\epsilon_n}\|_\infty > A$. Let $I$ denote the set consisting of all $\epsilon_n$, $n \geqslant N_A$. By the properties of $L^\infty$ norm, for each $n \geqslant N_A$ we can find $E_n$ a subset of $\Omega \times (0,T)$ with $0 < |E_n| < \frac{1}{2^n}$, such that $|w_{\epsilon_n}(x)| > A$ for a.e. $x \in E_n$. The set $E = \bigcup_{n \geqslant N_A} E_n$, satisfies $0 < |E| < \infty$ and $\|w_{\epsilon_n}\|_{\infty, E} > A$. Now the proof of (24) in Step 1 and the proofs in Steps 2 and 3 remain unchanged with the obvious modification that $\Upsilon$ is replaced by $I$ defined above. At the end of Step 4, we obtain that $\|\int_V |g_\epsilon| \, dv\|_{L^\infty(\Omega \times (0,T))}$ is uniformly in $\epsilon_n \in I$ bounded. This contradicts the fact that $\|w_{\epsilon_n}\|_\infty \to \infty$ as $n \to \infty$.

*Second Proof of the $L^\infty$ Bound*

Here, we shall assume that $|g^0_\epsilon(x, v)| \leqslant 1$, $|g_{\epsilon 0}(y, v, t)| \leqslant 1$, $|g_{\epsilon 1}(y, v, t)| \leqslant 1$. We shall also assume as in ref. 8 that $g^0_\epsilon(x, v) \operatorname{sign}(v) = |g^0_\epsilon(x, v)|$, $g_{\epsilon 0}(y, v, t) \operatorname{sign}(v) = |g_{\epsilon 0}(y, v, t)|$, and $g_{\epsilon 1}(y, v, t) \operatorname{sign}(v) = |g_{\epsilon 1}(y, v, t)|$. Let $\tilde{v}$ denote a positive number such that the support in $v$ of $g^0_\epsilon$, $g_{\epsilon 0}$, and $g_{\epsilon 1}$ is included in $[-\tilde{v}, \tilde{v}]$ (recall that we assumed that these data have supports that are included in a fixed compact set of $V$). Then using the sign condition on the data and the integral representation we conclude that $g_\epsilon(x, v, t) \operatorname{sign}(v) = |g_\epsilon(x, v, t)|$. Using the fact that the data are bounded by 1 and the integral representation respectively in $\Omega_0$, $\Omega_{01}$, and $\Omega_1$, we obtain that $|g_\epsilon(x, v, t)| \leqslant 1$.

To obtain the uniform in $\epsilon$ bound of $w_\epsilon$, we use the iterations (14)–(17) and its corresponding integral representation

In $\Omega_0$

$$g^{(1)}_\epsilon(x, v, t) = g_{\epsilon 0}\left(x_\star - \frac{x_1}{a_1(v)}\, a_\star(v), v, t - \frac{x_1}{a_1(v)}\right) \exp(-x_1/(a_1(v)\, \epsilon))$$
$$+ \frac{1}{\epsilon} \int_{t - \frac{x_1}{a_1(v)}}^{t} e^{(s-t)/\epsilon} \chi_{w^0_\epsilon(x(s), s)}(v)\, ds$$

In $\Omega_{01}$

$$g^{(1)}_\epsilon(x, v, t) = g^0_\epsilon(x - a(v)\, t, v) \exp(-t/\epsilon) + \frac{1}{\epsilon} \int_0^t e^{(s-t)/\epsilon} \chi_{w^0_\epsilon(x(s), s)}(v)\, ds$$

In $\Omega_1$

$$g_\epsilon^{(1)}(x, v, t) = g_{\epsilon 1}\left(x_\star + \frac{1-x_1}{a_1(v)}\, a_\star(v), v, t - \frac{x_1 - 1}{a_1(v)}\right)\exp((1-x_1)/\epsilon a_1(v))$$
$$+ \frac{1}{\epsilon}\int_{t-\frac{x_1-1}{a_1(v)}}^{t} e^{(s-t)/\epsilon}\chi_{w_\epsilon^0(x(s), s)}(v)\, ds$$

where $x(s) = x + (s-t)\, a(v)$, $x = (x_1, x_\star)$, and $a(v) = (a_1(v), a_\star(v))$.

Let $w^0$ be an initial iterate such that $\|w^0\|_{L^\infty(\Omega\times(0,T))} \leqslant \tilde{v}$. Then by definition of $\tilde{v}$, we have $g_{\epsilon 0}(y, v, t) = 0$, $g_{\epsilon 1}(y, v, t) = 0$, $g_\epsilon^0(x, v) = 0$, and $\chi_{w^0(x,t)}(v) = 0$, for all $v$ with $|v| > \tilde{v}$.

Now using the above integral representation, we conclude that $g_\epsilon(x, v, t) = 0$ for $|v| > \tilde{v}$. Using this and the sign property of $g_\epsilon$ ($|g_\epsilon(x, v, t)| = g_\epsilon(x, v, t)\,\text{sign}(v)$), we obtain

$$\begin{aligned}
|w_\epsilon^{(1)}(x, t)| &= \left|\int_V g_\epsilon(x, v, t)\, dv\right| \\
&\leqslant \max\left(\left|\int_{v>0} g_\epsilon(x, v, t)\, dv\right|, \left|\int_{v<0} |g_\epsilon(x, v, t)|\, dv\right|\right) \\
&\leqslant \tilde{v}
\end{aligned}$$

Thus, the contraction operator maps elements $w^{(0)}$ with $\|w^0\|_{L^\infty(\Omega\times(0,T))} < \tilde{v}$ into element with the same property. Therefore the fixed point $w_\epsilon$ has also this property. This concludes the proof of the uniform bound in $\epsilon$ of $w_\epsilon$ in $L^\infty(\Omega\times(0,T))$.

(ii) Now set $w_\infty = \sup_{\epsilon>0} \|w_\epsilon\|_{L^\infty(\Omega\times[0,T])}$, the terms $\chi_{w_\epsilon}$ in the integral representation in Theorem 2.1 are supported by $v \in [-w_\infty, w_\infty]$, the other terms are supported by $v$ in the compact supports of the boundary and initial data. Thus, for all $t \in [0, T]$, $g_\epsilon$ remains compactly supported, with compact supports included in $\text{Supp}_v\, g_\epsilon^0 \cup \text{Supp}_v\, g_{\epsilon 0} \cup \text{Supp}_v\, g_{\epsilon 1} \cup [-w_\infty, w_\infty]$, which in turn are included in a fixed compact set independent of $\epsilon$.

(iii) Now set $a_\infty = \sup_{1\leqslant i\leqslant N,\, v\in S} |a_i(v)|$, with $S = \text{Supp}_v\, g_\epsilon^0 \cup \text{Supp}_v\, g_{\epsilon 1} \cup \text{Supp}_v\, g_{\epsilon 0} \cup [-w_\infty, w_\infty]$. We conclude that $\sup_{1\leqslant i\leqslant N,\, v\in S'} |a_i(v)| \leqslant a_\infty$, where $S' = \{v \in \text{supp}_v\, g_\epsilon(x, ., t), (x, t) \in \Omega\times(0, T)\}$. And the lemma is proved.

In order to pass to the limit as the microscopic scale goes to 0, we shall need to control the spatial and temporal variations of $g_\epsilon$ and $w_\epsilon$ in terms of $\epsilon$. This is given in the following lemma.

**Lemma 2.4.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega\times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3,$$
$$\|g_\epsilon^0\|_{L^1(\Omega\times V)} < C_4, \qquad \|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^- \times (0,T))} < C_5,$$
$$\|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^- \times (0,T))} < C_6, \qquad \|g_\epsilon^0\|_{L^1(V;\, BV_{\mathrm{loc}}(\Omega))} < C_7,$$

with $C_i$, $i = 1,\dots, 7$ positive constants independent of $\epsilon$. Assume also that the initial and boundary data $f_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$.

Then

(1) $g_\epsilon(\cdot,\cdot,t)$ and $w_\epsilon(\cdot,t)$, $t\in[0,T]$ are uniformly bounded in $BV_{\mathrm{loc}}(\Omega\times L^1(V))$ and $BV_{\mathrm{loc}}(\Omega)$ respectively.

(2) $w_\epsilon$ is time Lipschitz continuous in $L^1_{\mathrm{loc}}(\Omega)$ uniformly in $\epsilon$; i.e., for any open bounded subset $U$ of $\Omega$ with $\bar U \subset \Omega$, we have

$$\begin{aligned}&\|w_\epsilon(\cdot,t_2) - w_\epsilon(\cdot,t_1)\|_{L^1(U)}\\ &\quad < a_\infty \|g_\epsilon\|_{L^\infty([0,T];\, BV(U\times L^1(V)))}(t_2-t_1) < C(t_2-t_1), \qquad \forall 0\leqslant t_1 < t_2 \leqslant T\end{aligned} \tag{35}$$

where $C$ is a constant depending on $U$ but is independent of $\epsilon$ and $a_\infty$ is introduced in the proof of Lemma 2.3 above.

(3) Under the additional assumption

$$\|g_\epsilon^0(\cdot,\cdot) - \chi_{w^0(\cdot)}(\cdot)\|_{L^1_{\mathrm{loc}}(\Omega\times L^1(V))} \xrightarrow[\epsilon\to 0]{} 0 \tag{36}$$

we can estimate the error between the kinetic solution and exact entropy solution as follows

$$\begin{aligned}&\|g_\epsilon - \chi_{w_\epsilon}\|_{L^\infty([0,T];\, L^1_{\mathrm{loc}}(\Omega\times L^1(V)))}\\ &\quad \leqslant \epsilon a_\infty \|g_\epsilon^0(x,v)\|_{BV_{\mathrm{loc}}(\Omega\times L^1(V))} + \epsilon a_\infty \|g_\epsilon(x,v,t)\|_{L^\infty([0,T];\, BV_{\mathrm{loc}}(\Omega\times L^1(V)))}\\ &\qquad + 2\|g_\epsilon^0(x,v) - \chi_{w^0(x)}\|_{L^1_{\mathrm{loc}}(\Omega\times L^1(V))} \xrightarrow[\epsilon\to 0]{} 0\end{aligned} \tag{37}$$

(4) The function $w_\epsilon$ is uniformly bounded in $BV_{\mathrm{loc}}(\Omega\times(0,T))$.

***Proof.*** (1) Let $0 < t < T$ be fixed and $h > 0$ be small. The case of $h < 0$ will be handled similarly. Let $\tau_h^i g_\epsilon(x,v,t) = g_\epsilon(x_1,\dots, x_i + he_i,\dots, x_d, v, t)$, $i = 1,\dots, d$. Multiplying the equation (1) for $\tau_h^1 g_\epsilon - g_\epsilon$ by $\varphi$ with $\varphi$ a test function which is Lipschitz continuous in $(0,1-h)\times\mathbb{R}^{d-1}\times V\times[0,T]$ with compact support in $x$ in $(0,1-h)\times\mathbb{R}^{d-1}$ to be precised later, and integrating by parts, we obtain

$$\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V}((\tau_h^1 g_\epsilon - g_\epsilon)\,\varphi)(\cdot,\cdot,t) - \int_{(0,1-h)\times\mathbb{R}^{d-1}\times V}((\tau_h^1 g_\epsilon - g_\epsilon)\,\varphi)(\cdot,\cdot,0)$$
$$-\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V\times(0,t)}(\partial_t\varphi + a(v)\cdot\partial_x\varphi)(\tau_h^1 g_\epsilon - g_\epsilon)$$
$$= \frac{1}{\epsilon}\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V\times(0,t)}((\chi_{\tau_h^1 w_\epsilon} - \chi_{w_\epsilon}) - (\tau_h^1 g_\epsilon - g_\epsilon))\,\varphi \tag{38}$$

We then take $\varphi = \text{sign}^\mu(\tau_h^1 g_\epsilon - g_\epsilon)\,\psi(x,t)$ with $x\,\text{sign}^\mu(x) \geqslant 0\ \ x\in\mathbb{R}$, and $\psi$ is a nonnegative test function which is Lipschitz continuous in $(0,1-h)\times\mathbb{R}^{d-1}\times V\times[0,T]$ with compact support in $x$ in $(0,1-h)\times\mathbb{R}^{d-1}$ and $\text{sign}^\mu$ is a regularization of sign function. Proceeding as in the proof of Theorem 2.1, we obtain

$$\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V}|\tau_h^1 g_\epsilon - g_\epsilon|\,\psi(\cdot,\cdot,t) - \int_{(0,1-h)\times\mathbb{R}^{d-1}\times V}|\tau_h^1 g_\epsilon - g_\epsilon|\,\psi(\cdot,\cdot,0)$$
$$-\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V\times(0,t)}(\partial_t\psi + a(v)\cdot\partial_x\psi)\,|\tau_h^1 g_\epsilon - g_\epsilon|$$
$$= \frac{1}{\epsilon}\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V\times(0,t)}((\chi_{\tau_h^1 w_\epsilon} - \chi_{w_\epsilon}) - (\tau_h^1 g_\epsilon - g_\epsilon))\,\text{sign}(\tau_h^1 g_\epsilon - g_\epsilon)\,\psi$$
$$\leqslant 0$$

where in the last inequality we have used the properties of $\chi$, we then have

$$\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V}\psi\,|\tau_h^1 g_\epsilon - g_\epsilon|\,(\cdot,\cdot,t)$$
$$\leqslant \int_{(0,1-h)\times\mathbb{R}^{d-1}\times V}\psi\,|\tau_h^1 g_\epsilon - g_\epsilon|\,(\cdot,\cdot,0)$$
$$+\int_{(0,1-h)\times\mathbb{R}^{d-1}\times V\times(0,t)}(\partial_t\psi + a(v)\cdot\partial_x\psi)\,|\tau_h^1 g_\epsilon - g_\epsilon|$$

In particular we have

$$\int_{O\times V}\psi\,|\tau_h^1 g_\epsilon - g_\epsilon|\,(\cdot,\cdot,t)$$
$$\leqslant \int_{O\times V}\psi\,|\tau_h^1 g_\epsilon - g_\epsilon|\,(\cdot,\cdot,0) + \int_{O\times V\times(0,t)}(\partial_t\psi + a(v)\cdot\partial_x\psi)\,|\tau_h^1 g_\epsilon - g_\epsilon| \tag{39}$$

for any open set with $\bar{O} \subset (0, 1-h) \times \mathbb{R}^{d-1}$ and $\psi$ any Lipschitz continuous function in $O \times V \times [0, T]$ with compact support in $x$ in $O$. Similarly, we have for $i = 2,..., d$

$$\int_{O \times V} \psi \, |\tau_h^i g_\epsilon - g_\epsilon| \, (\cdot, \cdot, t)$$
$$\leqslant \int_{O \times V} \psi \, |\tau_h^i g_\epsilon - g_\epsilon| \, (\cdot, \cdot, 0) + \int_{O \times V \times (0, t)} (\partial_t \psi + a(v) \cdot \partial_x \psi) \, |\tau_h^i g_\epsilon - g_\epsilon| \tag{40}$$

for any open set with $\bar{O} \subset (0, 1) \times \mathbb{R}^{d-1}$ and $\psi$ any Lipschitz continuous function in $O \times V \times [0, T]$ with compact support in $x$ in $O$.

Let $i \in \{2,..., d\}$ be fixed. Let $U$ and $O$ be open bounded subsets of $\Omega$ such that $\bar{U} \subset O \subset \bar{O} \subset \Omega$. Let $\psi$ be a Lipschitz continuous function in $O \times V \times [0, T]$ with compact support in $x$ in $O$ such that $U \subset \mathrm{supp}_x \, \psi \subset O$. Then (40) holds for such $\psi$ and $O$.

We wish to prove that

$$\int_{U \times V} |\tau_h^i g_\epsilon - g_\epsilon| \leqslant Ch \tag{41}$$

where $C$ depends on $U$ but is independent of $\epsilon$. It is enough to prove this relation for $U$ of the form $U = (y_1 - \alpha, y_1 + \alpha) \times B(y_\star, R)$ where $\alpha > 0$ and $y = (y_1, y_\star) \in \Omega$ are arbitrary elements of $\mathbb{R}^{+\star}$ and $\Omega$ such that $0 < y_1 - \alpha < y_1 + \alpha < 1$ and $R > 0$ is arbitrary radius. Let $\beta > 0$ and $\gamma > 0$ be such that $0 < y_1 - \alpha - \beta - \gamma < y_1 + \alpha + \beta + \gamma < 1$. Let $0 < t_1 < T$ be such that $a_\infty t_1 = \beta$. Let $O = (y_1 - \alpha - a_\infty t_1 - \gamma, y_1 + \alpha + a_\infty t_1 + \gamma) \times B(y_\star, R + \delta + d a_\infty t_1)$, with $\delta > 0$. Let $t \in (0, t_1]$. Consider now the functions

$$\varphi_1(x_1, \tau) = \begin{cases} 0 & 0 \leqslant x_1 < y_1 - \alpha - a_\infty(t-\tau) - \gamma \\ & 0 \leqslant \tau \leqslant t \\ \frac{1}{\gamma}(x_1 - y_1 + \alpha + a_\infty(t-\tau)) + 1 & y_1 - \alpha - a_\infty(t-\tau) - \gamma \leqslant x_1 < \\ & y_1 - \alpha - a_\infty(t-\tau), \quad 0 \leqslant \tau \leqslant t \\ 1 & y_1 - \alpha - a_\infty(t-\tau) \leqslant x_1 < \\ & y_1 + \alpha + a_\infty(t-\tau) \quad 0 \leqslant \tau \leqslant t \\ \frac{1}{\gamma}(y_1 + \alpha + a_\infty(t-\tau) - x_1) + 1 & y_1 + \alpha + a_\infty(t-\tau) \leqslant x_1 < \\ & y_1 + \alpha + a_\infty(t-\tau) + \gamma, \quad 0 \leqslant \tau \leqslant t \\ 0 & y_1 + \alpha + a_\infty(t-\tau) + \gamma \leqslant x_1 \leqslant 1 \\ & 0 \leqslant \tau \leqslant t \end{cases}$$

and

$$\varphi_2(x_\star,\tau)=\begin{cases}1 & 0\leqslant |x_\star-y_\star|<R+da_\infty(t-\tau)\\ & 0\leqslant\tau\leqslant t\\ \frac{1}{\delta}(R+da_\infty(t-\tau)-|x_\star-y_\star|)+1 & R+da_\infty(t-\tau)\leqslant|x_\star-y_\star|<\\ & R+da_\infty(t-\tau)+\delta,\quad 0\leqslant\tau\leqslant t\\ 0 & R+da_\infty(t-\tau)+\delta\leqslant|x_\star-y_\star|\\ & 0\leqslant\tau\leqslant t\end{cases}$$

Now let $\psi(x,\tau)=\varphi_1(x_1,\tau)\,\varphi_2(x_\star,\tau)$, $\tau\in[0,t]$ and $x=(x_1,x_\star)$. It is clear that $\psi$ is nonnegative Lipschitz continuous function in $O\times V\times[0,t]$ with compact support in $x$ in $O$ and $U\subset \operatorname{supp}_x\psi\subset O$. Thus, plugging $\psi$ in (40) and using the fact that $g_\epsilon^0$ is uniformly bounded in $BV_{\text{loc}}(\Omega\times L^1(V))$ (since $g_\epsilon^0$ is uniformly bounded in $L^1(V;BV_{\text{loc}}(\Omega))\subset BV_{\text{loc}}(\Omega\times L^1(V))$) yields (41) for $t\in(0,t_1]$. Now let $t_2>t_1$ be such that $a_\infty(t_2-t_1)=\beta$. Proceeding as above and using the fact that $g_\epsilon(\cdot,\cdot,t_1)$ is uniformly bounded in $BV_{\text{loc}}(\Omega\times L^1(V))$, we conclude that $g_\epsilon(\cdot,\cdot,t)$ is uniformly bounded in $BV_{\text{loc}}(\Omega\times L^1(V))$ for any $t\in(t_1,t_2]$. Continuing this process we conclude that $g_\epsilon(\cdot,\cdot,t)$ is uniformly bounded in $BV_{\text{loc}}(\Omega\times L^1(V))$ for any $t\in[0,T]$.

Finally, using similar constructions we can prove that for any open bounded subset $O$ of $(0,1-h)\times\mathbb{R}^{d-1}$ with $\bar{O}\subset(0,1-h)\times\mathbb{R}^{d-1}$, we have

$$\int_{O\times V}|\tau_h^1 g_\epsilon-g_\epsilon|\leqslant Ch$$

where $C$ is a positive constant depending on $O$, but is independent of $\epsilon$. This concludes the proof that $g_\epsilon$ is uniformly bounded in $L^\infty([0,T];BV_{\text{loc}}(\Omega\times L^1(V)))$. The uniform bound of $w_\epsilon$ in $L^\infty([0,T];BV_{\text{loc}}(\Omega))$ can then be deduced from that of $g_\epsilon$. And the statement 1) is proved.

(2) Let $0\leqslant t_1<t_2\leqslant T$ and $U$ be an open bounded subset of $\Omega$ with $\bar{U}\subset\Omega$. Let $\psi(x)\in C_0^1(U)$. Multiplying Eq. (1) by $\psi$ and integrating over $U\times(t_1,t_2)\times V$, we obtain

$$\int_{U\times V\times(t_1,t_2)}\partial_t g_\epsilon\psi+\sum_i\int_{U\times V\times(t_1,t_2)}a_i(v)\,\partial_{x_i}g_\epsilon\psi=\frac{1}{\epsilon}\int_{U\times V\times(t_1,t_2)}(\chi_{w_\epsilon}-g_\epsilon)\,\psi=0$$

Hence, we have

$$\int_U(w_\epsilon(x,t_2)-w_\epsilon(x,t_1))\,\psi(x)=-\int_{t_1}^{t_2}\sum_i\int_{U\times V}a_i(v)\,\partial_{x_i}g_\epsilon\psi \tag{42}$$

Since $\partial_{x_i} g_\epsilon$, $i=1,\ldots,d$ are locally finite measures (consult (1) above), the integrand on the right side is bounded by $a_\infty C(U)$ for $|\psi(x)| \leqslant 1$. Taking the supremum of (42) over all $\psi$ with $|\psi(x)| \leqslant 1$ yields (35).

(3) Let $U$ be an open bounded set of $\Omega$ such that $\bar{U} \subset \Omega$. Let $O$ be an open bounded set of $\Omega$ such that $\bar{U} \subset O \subset \bar{O} \subset \Omega$. Taking $G_\epsilon(x, v, t) = g_\epsilon(x, v, t+\Delta t)$ and proceeding as in the derivation of (10) and the proof of the uniform $BV_{\text{loc}}$ bound (consult part (1) above), we obtain

$$\int_{U\times V} |g_\epsilon(x, v, t+\Delta t) - g_\epsilon(x, v, t)| \leqslant \int_{O\times V} |g_\epsilon(x, v, \Delta t) - g_\epsilon(x, v, 0)| \tag{43}$$

from which we deduce

$$\|\partial_t g_\epsilon(x, v, t)\|_{L^1(U\times V)} \leqslant \|\partial_t g_\epsilon(x, v, t=0)\|_{L^1(O\times V)} \tag{44}$$

The kinetic equation (1) yields

$$\begin{aligned} &\|\partial_t g_\epsilon(x, v, t=0)\|_{L^1(O\times V)} \\ &\quad\leqslant \|(a(v)\cdot\partial_x)\, g_\epsilon(x, v, t=0)\|_{L^1(O\times V)} + \frac{1}{\epsilon}\, \|\chi_{w_\epsilon(x,\, t=0)} - g_\epsilon(x, v, t=0)\|_{L^1(O\times V)} \\ &\quad\leqslant a_\infty\, \|g_\epsilon(x, v, t=0)\|_{BV(O\times L^1(V))} + \frac{2}{\epsilon}\, \|g_\epsilon^0(x, v) - \chi_{w^0(x)}\|_{L^1(O\times V)} \end{aligned} \tag{45}$$

Using again the kinetic equation (1) together with (44), (45) and the uniform bound of $g_\epsilon(x, v, t)$ in $L^\infty([0, T], BV_{\text{loc}}(\Omega \times L^1(V)))$, we obtain

$$\begin{aligned} &\|g_\epsilon(x, v, t) - \chi_{w_\epsilon(x,t)}(v)\|_{L^1(U\times V)} \\ &\quad\leqslant \epsilon\, \|\partial_t g_\epsilon(x, v, t)\|_{L^1(U\times V)} + \epsilon\, \|(a(v)\cdot\partial_x)\, g_\epsilon(x, v, t)\|_{L^1(U\times V)} \\ &\quad\leqslant \epsilon a_\infty\, \|g_\epsilon(x, v, t=0)\|_{BV(O\times L^1(V))} + \epsilon a_\infty\, \|g_\epsilon(x, v, t)\|_{BV(U\times L^1(V))} \\ &\qquad + 2\, \|g_\epsilon^0(x, v) - \chi_{w^0(x)}\|_{L^1(O\times V)} \end{aligned} \tag{46}$$

Now, (46) and (36) yield as $\epsilon \to 0$

$$\|g_\epsilon(x, v, t) - \chi_{w_\epsilon(x,t)}(v)\|_{L^1(U\times V)} \to 0$$

The proof of (3) is now complete.

(4) The proof is an immediate consequence of a combination of (1) and (2) above.

**Remark 2.4.** Notice that Lemma 2.4(1) furnishes a local uniform in $\epsilon$ bound on the spatial variation on the microscopic scale. However, the local Lipschitz continuity is obtained only at the macroscopic level; consult Lemma 2.4(2). The temporal variation at the microscopic level cannot, in general, be bounded uniformly in $\epsilon$. Such uniform control can, however, be achieved if we assume that the initial data $g_\epsilon^{(0)}$ converges to the equilibrium $\chi_{w^0}$ in $L^1_{\mathrm{loc}}(\Omega \times L^1(V))$ (consult Theorem 3.3 and the remark before it).

## 3. HYDRODYNAMIC LIMIT OF THE KINETIC PROBLEM AND EXISTENCE THEORY FOR THE CONSERVATION LAWS

In this section we shall prove that the conservation laws (5)–(7) has a solution in the sense of Definition 3.1 below which selects a physically correct solution to this problem.

**Definition 3.1.** We say that $w \in BV_{\mathrm{loc}}(\Omega \times (0, T)) \cap L^\infty(\Omega \times [0, T])$ is a weak entropic solution of the problem (5)–(7) if we have

$$
\begin{aligned}
&-\int_{\Omega \times (0,T)} (|w-k|\, \partial_t \psi + \mathrm{sign}(w-k)(A(w)-A(k)) \cdot \nabla_x \psi) \\
&\quad + \int_{\Gamma_1 \times (0,T)} \psi\, \mathrm{sign}(w_1 - k)((A(w_1)\cdot n)^- - (A(k)\cdot n)^-) \\
&\quad + \int_{\Gamma_0^- \times (0,T)} a(v)\cdot n \psi\, |g_0 - \chi_k| \leqslant 0 \\
&\qquad \forall \psi \in C_0^1(\bar{\Omega} \times V \times (0,T)), \quad \psi \geqslant 0, \quad \forall k \in \mathbb{R}
\end{aligned}
$$

and $w$ satisfies the initial condition

$$
w(x,0) = w^0(x) \qquad \text{in } \Omega
$$

We now state the following theorem about the existence of a solution to the conservation laws.

**Theorem 3.1.** Assume that

$$
\begin{gathered}
\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega \times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3, \\
\|g_\epsilon^0\|_{L^1(\Omega \times V)} < C_4, \qquad \|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^- \times (0,T))} < C_5, \\
\|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^- \times (0,T))} < C_6, \qquad \|g_\epsilon^0\|_{L^1(V;\, BV_{\mathrm{loc}}(\Omega))} < C_7
\end{gathered}
$$

with $C_i$, $i = 1,\ldots, 7$ positive constants independent of $\epsilon$.

Assume also that the initial and boundary data $f_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$. Finally assume that as $\epsilon \to 0$,

$$\|w_\epsilon(\cdot, 0) - w^0(\cdot)\|_{L^1_{\mathrm{loc}}(\Omega)} = \left\| \int_V g_\epsilon^0(\cdot, v) - w^0(\cdot) \right\|_{L^1_{\mathrm{loc}}(\Omega)} \to 0 \tag{47}$$

$$a(v) \cdot n g_{\epsilon 0} \to a(v) \cdot n g_0 \qquad \text{strongly in} \quad L^1(\Gamma_0^- \times (0, T)) \tag{48}$$

$$a(v) \cdot n g_{\epsilon 1} \to a(v) \cdot n g_1 = a(v) \cdot n \chi_{w_1} \qquad \text{strongly in} \quad L^1(\Gamma_1^- \times (0, T)) \tag{49}$$

Then $w_\epsilon$ converges strongly in $L^1(\Omega \times (0, T))$, as $\epsilon$ goes to 0, to an entropic solution of the problem (5)–(7) in the sense of Definition 3.1.

Before we give the proof of Theorem 3.1, we shall state and prove a preliminary result showing compactness of $w_\epsilon$ and $g_\epsilon$ respectively in $L^1(\Omega \times (0, T))$ and $L^1(\Omega \times V \times (0, T))$. We shall assume that $\Omega = (0, 1)$. It is not difficult to generalize our proof to the case $\Omega = (0, 1) \times \mathbb{R}^{d-1}$.

**Lemma 3.1.** Assume that all assumptions of Theorem 3.1 hold. Then

(i) A subsequence of $w_\epsilon$ (still denoted $w_\epsilon$) converges as $\epsilon \to 0$ to $w$ in $L^1_{\mathrm{loc}}(\Omega \times (0, T)) \cap L^\infty([0, T]; L^1_{\mathrm{loc}}(\Omega))$ and in $L^\infty(\Omega \times [0, T])$ weak-★. Moreover $w_\epsilon$ converges a.e. to $w$ in $\Omega \times (0, T)$ and $w \in BV_{\mathrm{loc}}(\Omega \times (0, T))$.

(ii) The $L^1_{\mathrm{loc}}$ convergence of $w_\epsilon$ takes place actually in $L^1(\Omega \times (0, T)) \cap L^\infty([0, T]; L^1(\Omega))$.

(iii) Finally, we have $\|g_\epsilon - \chi_w\|_{L^1(\Omega \times V \times (0, T))} \to 0$ as $\epsilon \to 0$.

To prove Lemma 3.1(ii), we shall also need the following result.

**Theorem 3.2.** Let $U$ be a bounded open subset of $\mathbb{R}^N$ and let $v_n$ be a sequence in $L^1_{\mathrm{loc}}(U)$. Assume that as $n \to \infty$, the sequence $v_n$ converges strongly in $L^1_{\mathrm{loc}}(U)$ to $v \in L^1_{\mathrm{loc}}(U)$. If $v_n$ is uniformly bounded in $L^\infty(U)$ then $v_n$ converges strongly to $v$ in $L^1(U)$.

*Proof of Theorem 3.2.* Let $\eta > 0$ be fixed. Since $U$ is bounded there exists a compact set $K_\eta \subset U$ such that the Lebesgue measure meas$(U \setminus K_\eta) < \eta$. On the other hand since $v_n$ is uniformly bounded in $L^\infty(U)$, by diagonal process to pass to a further subsequence if necessary and uniqueness of the limit, $v_n$ converges in $L^\infty$ weak-★ to $v$. Hence $v \in L^\infty(U)$. Now

$$\begin{aligned}\int_U |v_n - v| &= \int_{U\setminus K_\eta} |v_n - v| + \int_{K_\eta} |v_n - v| \\ &\leqslant \|v_n - v\|_\infty \operatorname{meas}(U\setminus K_\eta) + \int_{K_\eta} |v_n - v| \\ &\leqslant C\eta + \int_{K_\eta} |v_n - v|\end{aligned}$$

where $C$ is a constant independent of $n$ and $\eta$. Therefore since $\lim_{n\to\infty} \int_{K_\eta} |v_n - v| = 0$,

$$\limsup_{n\to\infty} \int_U |v_n - v| \leqslant C\eta$$

This proves the statement since $\eta$ is arbitrary.

*Proof of Lemma 3.1.* Using Lemma 2.4(4) and Lemma 2.2 $w_\epsilon$ is bounded uniformly in $L^1 \cap BV_{\text{loc}}(\Omega\times(0,T))$. Hence a subsequence of $w_\epsilon$ (still denoted $w_\epsilon$) converges to $w$ in $L^1_{\text{loc}}(\Omega\times(0,T))$ and almost everywhere in $\Omega\times(0,T)$. Moreover $w \in BV_{\text{loc}}(\Omega\times(0,T))$. Using Lemma 2.3 and diagonal process to extract a further subsequence, if necessary, $w_\epsilon$ converges in $L^\infty(\Omega\times[0,T])$ weak-★ to a function $w\in L^\infty(\Omega\times[0,T])$. Since $\Omega\times(0,T)$ is bounded the limit $w$ is in $L^1(\Omega\times(0,T))$. Now by the dominated convergence theorem and the above, the convergence of $w_\epsilon$ takes place in fact in $L^1(\Omega\times(0,T))$.

Now by Lemma 2.4(1) $w_\epsilon(\cdot,t), t\in[0,T]$ is uniformly bounded in $BV_{\text{loc}}(\Omega)$. Hence it is precompact in $L^1_{\text{loc}}(\Omega)$. Using Lemma 2.4(2), $\|w_\epsilon(x,t)\|_{L^1(\Omega)}$ is Lipschitz continuous in time. By diagonal process to extract a further subsequence, if necessary, $w_\epsilon \xrightarrow[\epsilon\to 0]{} w$ strongly in $L^\infty([0,T];L^1_{\text{loc}}(\Omega))$. Now by the same process we used to prove the strong $L^1$ convergence of $w_\epsilon$ to $w$ in $L^1(\Omega\times(0,T))$, we conclude that $w_\epsilon \xrightarrow[\epsilon\to 0]{} w$ strongly in $L^\infty([0,T];L^1(\Omega))$.

By the properties of $\chi$, we conclude that $\chi_{w_\epsilon}$ strongly converges to $\chi_w$ in $L^1$. Using this and the integral representation (Theorem 2.1), and recalling that the boundary data satisfy (48)–(49), we infer that $g_\epsilon$ strongly converges to $\chi_w$ in $L^1$. This concludes the proof of the lemma.

*Proof of Theorem 3.1.* Using Lemma 3.1 a subsequence of $w_\epsilon$ (still denoted $w_\epsilon$) converges strongly in $L^1$ to $w$. We know that $w\in BV_{\text{loc}}(\Omega\times(0,T))\cap L^\infty(\Omega\times[0,T])$ (consult the proof of Lemma 3.1). Using Theorem 2.2, we have

$$-\int_{\Omega\times V\times(0,T)} (\partial_t + a(v)\cdot\partial_x)(\psi)\,|g_\epsilon-\chi_k| + \int_{\Gamma_0^-\times(0,T)} a(v)\cdot n\psi\,|g_{\epsilon 0}-\chi_k|$$

$$+\int_{\Gamma_1^-\times(0,T)} a(v)\cdot n\psi\,|g_{\epsilon 1}-\chi_k|$$

$$\leqslant 0 \qquad \forall\psi\in C_0^1(\bar\Omega\times(0,T)),\quad \psi\geqslant 0,\quad \forall k\in\mathbb{R}$$

Using Lemmas 3.1 and 2.3, (48) and (49), and the properties of $\chi$, we then obtain

$$-\int_{\Omega\times(0,T)} \partial_t\psi\,|w-k| - \int_{\Omega\times(0,T)} \operatorname{sign}(w-k)(A(w)-A(k))\cdot\partial_x\psi$$

$$+\int_{\Gamma_0^-\times(0,T)} a(v)\cdot n\psi\,|g_0-\chi_k|$$

$$+\int_{\Gamma_1\times(0,T)} \operatorname{sign}(w_1-k)((A(w_1)\cdot n)^- - (A(k)\cdot n)^-)\,\psi$$

$$\leqslant 0 \qquad \forall\psi\in C_0^1(\bar\Omega_g\times(0,T)),\quad \psi\geqslant 0,\quad \forall k\in\mathbb{R} \tag{50}$$

Finally, thanks to Lemma 3.1 and (47), $w$ satisfies the initial conditions (7). Thus, combining (50) and the above, it is clear that $w$ is an entropic solution in the sense of Definition 3.1 to the problem (5)–(7).

The proof of the theorem is now complete.

As we saw in Remark 2.4, the temporal variation at the microscopic level cannot, in general, be bounded uniformly in $\epsilon$. Such uniform control can be achieved if we assume that the kinetic initial data satisfies

$$\|g_\epsilon^0(\cdot,\cdot)-\chi_{w^0(\cdot)}(\cdot)\|_{L^1_{\mathrm{loc}}(\Omega\times L^1(V))} \xrightarrow[\epsilon\to 0]{} 0$$

**Theorem 3.3.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^-\times[0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega\times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^-\times[0,T])} < C_3,$$

$$\|g_\epsilon^0\|_{L^1(\Omega\times V)} < C_4, \qquad \|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^-\times(0,T))} < C_5,$$

$$\|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^-\times(0,T))} < C_6, \qquad \|g_\epsilon^0\|_{L^1(V;\,BV_{\mathrm{loc}}(\Omega))} < C_7$$

with $C_i$, $i=1,\ldots,7$ positive constants independent of $\epsilon$.

Assume also that the initial and boundary data $f_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$. Finally assume that as $\epsilon \to 0$,

$$\|g_\epsilon^0(\cdot,\cdot) - \chi_{w^0(\cdot)}(\cdot)\|_{L^1_{\text{loc}}(\Omega \times L^1(V))} \xrightarrow[\epsilon \to 0]{} 0 \tag{51}$$

$$a(v) \cdot n g_{\epsilon 0} \to a(v) \cdot n g_0 \qquad \text{strongly in} \quad L^1(\Gamma_0^- \times (0,T)) \tag{52}$$

$$a(v) \cdot n g_{\epsilon 1} \to a(v) \cdot n g_1 = a(v) \cdot n \chi_{w_1} \qquad \text{strongly in} \quad L^1(\Gamma_1^- \times (0,T)) \tag{53}$$

Then $g_\epsilon$ converges strongly in $L^\infty([0,T]; L^1(\Omega \times V))$, as $\epsilon$ goes to 0, to $\chi_w$ and $w$ is an entropic solution of the problem (5)–(7) in the sense of Definition 3.1.

Before we give the proof of Theorem 3.3, we shall prove the lemma below.

**Lemma 3.2.** Assume that all assumptions of Theorem 3.3 hold. Then (i) and (ii) of Lemma 3.1 hold true. Moreover, we have $\|g_\epsilon - \chi_w\|_{L^\infty([0,T]; L^1(\Omega \times V))} \to 0$ as $\epsilon \to 0$.

*Proof of Lemma 3.2.* We only need to prove the last statement in the lemma. By Lemma 2.4(3)

$$\|g_\epsilon - \chi_{w_\epsilon}\|_{L^\infty([0,T]; L^1_{\text{loc}}(\Omega \times L^1(V)))} \xrightarrow[\epsilon \to 0]{} 0$$

Thus

$$\|g_\epsilon - \chi_w\|_{L^\infty([0,T]; L^1_{\text{loc}}(\Omega \times L^1(V)))} \xrightarrow[\epsilon \to 0]{} 0$$

Since $g_\epsilon$ is uniformly bounded in $L^\infty(\Omega \times V \times [0,T])$ (Lemma 2.1) and remains compactly supported in $v$ with support included in a fixed compact set independent of $\epsilon$ (Lemma 2.3), and $g_\epsilon$ converges to $g$ in $L^\infty([0,T]; L^1_{\text{loc}}(\Omega \times L^1(V)))$, we can apply Theorem 3.2 to infer that $g_\epsilon \to \chi_w$ in $L^\infty([0,T]; L^1(\Omega \times L^1(V)))$. This concludes the proof of the lemma.

*Proof of Theorem 3.3.* The proof of this theorem is similar to that of Theorem 3.1 and will not be repeated.

**Remark 3.1.** Theorems 3.1 and 3.3 are obtained under various assumptions including the assumptions that the data $g_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v$. In fact these theorems are also valid when these data are not necessarily compactly supported in $v$. The proof is based on a $BV$-regularization argument.

## 4. CANCELLATION OF MICROSCOPIC OSCILLATIONS VIA THE COMPENSATED COMPACTNESS

In this section we study the one-dimensional scalar conservation law

$$\partial_t w + \partial_x A(w) = 0 \qquad \text{in} \quad \Omega \times (0, T) \tag{54}$$

$$\text{Boundary conditions for } w \text{ on } \Gamma_0 \times (0, T) \text{ and } \Gamma_1 \times (0, T) \tag{55}$$

$$w(x, 0) = w^0(x) \qquad \text{in } \Omega \tag{56}$$

The corresponding kinetic equation is

$$[\partial_t + a(v) \cdot \partial_x]\, g_\epsilon(x, v, t) = \frac{1}{\epsilon} (\chi_{w_\epsilon(x,t)}(v) - g_\epsilon(x, v, t)) \qquad \text{in} \quad \Omega \times V \times (0, T) \tag{57}$$

$$g_\epsilon(x, v, t) = g_{\epsilon 0}(x, v, t) \qquad \text{on} \quad \Gamma_0^- \times (0, T) \tag{58}$$

$$g_\epsilon(x, v, t) = g_{\epsilon 1}(x, v, t) \qquad \text{on} \quad \Gamma_1^- \times (0, T), \tag{59}$$

$$g_\epsilon(x, v, 0) = g_\epsilon^0(x, v) \qquad \text{in} \quad \Omega \times V \tag{60}$$

where all data and the relationships between the various quantities above were precised in the introduction, we only need to take $d = 1$. We assume that the conservation law (54) is nonlinear in the sense that there exists no interval on which the flux $A(u)$ is linear, i.e., $A''(u) \neq 0$ a.e. In the full space case i.e., $\Omega = \mathbb{R}$, the study of this problem without using compactness arguments (based on $BV$ estimates as in Lemma 2.4) has been done in ref. 7. The authors use compensated compactness, specifically, the Tartar's div-curl lemma.[9] We shall extend this result to the case of domains with boundaries. We first give a definition of a solution to the nonlinear conservation laws.

**Definition 4.1.** We say that $w \in L^\infty(\Omega \times [0, T])$ is a weak entropic solution of the problem (54)–(56) if we have

$$-\int_{\Omega \times (0,T)} (|w-k|\, \partial_t \psi + \text{sign}(w-k)(A(w) - A(k)) \cdot \nabla_x \psi)$$

$$+ \int_{\Gamma_1 \times (0,T)} \psi \, \text{sign}(w_1 - k)((A(w_1) \cdot n)^- - (A(k) \cdot n)^-)$$

$$+ \int_{\Gamma_0^- \times (0,T)} a(v) \cdot n \psi \, |g_0 - \chi_k| \leqslant 0$$

$$\forall \psi \in C_0^1(\bar{\Omega} \times V \times (0, T)), \quad \psi \geqslant 0, \quad \forall k \in \mathbb{R}$$

and $w$ satisfies the initial condition

$$w(x, 0) = w^0(x) \qquad \text{in } \Omega$$

The main result of this section is

**Theorem 4.1.** Assume that the conservation law (54) is nonlinear (see above). Let $g_\epsilon$ be the solution of the corresponding kinetic equation (57)–(60). Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega \times V)} < C_2,$$
$$\|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3, \qquad \|g_\epsilon^0\|_{L^1(\Omega \times V)} < C_4,$$
$$\|a(v) \cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^- \times (0,T))} < C_5, \qquad \|a(v) \cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^- \times (0,T))} < C_6$$

with $C_i$, $i = 1,\ldots, 6$ positive constants independent of $\epsilon$.

Assume also that the initial and boundary data $g_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$. Finally assume that as $\epsilon \to 0$,

$$\|w_\epsilon(\cdot, 0) - w^0(\cdot)\|_{L^1_{\text{loc}}(\Omega)} = \left\| \int_V g_\epsilon^0(\cdot, v) - w^0(\cdot) \right\|_{L^1_{\text{loc}}(\Omega)} \to 0 \tag{61}$$

$$a(v) \cdot n g_{\epsilon 0} \to a(v) \cdot n g_0 \qquad \text{strongly in} \quad L^1(\Gamma_0^- \times (0, T)) \tag{62}$$

$$a(v) \cdot n g_{\epsilon 1} \to a(v) \cdot n g_1 = a(v) \cdot n \chi_{w_1} \qquad \text{strongly in} \quad L^1(\Gamma_1^- \times (0, T)) \tag{63}$$

Then $w_\epsilon = \int_V g_\epsilon(x, v, t)\, dv$ converges strongly in $L^p(\Omega \times (0, T))$, $p < \infty$, to an entropic solution of the nonlinear conservation law (54)–(56) in the sense of Definition 4.1.

**Remark 4.1.** (1) We observe that under the assumptions of the theorem above, the conclusions of Lemmas 2.1–2.3 remain valid.

(2) Remark 3.1 is also valid for Theorem 4.1.

*Proof of Theorem 4.1.* The proof follows the same lines as the one corresponding to the full space case in ref. 7. Thus, proceeding as in ref. 7, we obtain

$$\overline{\int_V a(v)\, g_\epsilon\, dv} = \overline{\int_V a(v)\, \chi_{w_\epsilon}\, dv} = \overline{A(w_\epsilon)} \tag{64}$$

$$\overline{A(w_\epsilon)} = A(\overline{w_\epsilon}) \tag{65}$$

for otherwise, $\overline{|w_\epsilon - \overline{w_\epsilon}|}(x, t) = 0$, which in turn yields again (65). Combining (64) and (65), and passing to the limit weakly in (57), we obtain

$$\frac{\partial}{\partial t}\overline{w_\epsilon} + \frac{\partial}{\partial x} A(\overline{w_\epsilon}) = 0$$

Hence a subsequence of $w_\epsilon$ (still denoted $w_\epsilon$) converges to a weak solution of the conservation law (54). Thanks to the nonlinearity of $A(w)$ and equality (65), we can use Tartar Theorem [ref. 9, Theorem 26] to conclude that $w_\epsilon$ strongly converges in $L^p_{\text{loc}}(\Omega \times (0, T))$, $1 \leqslant p < \infty$. This combined with the process used to prove Theorem 3.1 completes the proof of the theorem.

## 5. CONSERVATION LAWS WITH SOURCE TERMS

In this section we introduce the following kinetic model with forces

$$[\partial_t + a(v) \cdot \partial_x + S(x, t, v) \cdot \partial_v]\, g_\epsilon(x, v, t)$$

$$= \frac{1}{\epsilon}(\chi_{w_\epsilon(x, t)}(v) - g_\epsilon(x, v, t)) \qquad \text{in} \quad \Omega \times V \times (0, T) \tag{66}$$

$$g_\epsilon(x, v, t) = g_{\epsilon 0}(x, v, t) \qquad \text{on} \quad \Gamma_0^- \times (0, T) \tag{67}$$

$$g_\epsilon(x, v, t) = g_{\epsilon 1}(x, v, t) \qquad \text{on} \quad \Gamma_1^- \times (0, T), \tag{68}$$

$$g_\epsilon(x, v, 0) = g_\epsilon^0(x, v) \qquad \text{in} \quad \Omega \times V \tag{69}$$

and study its relation to the inhomogeneous scalar conservation laws

$$\partial_t w(x, t) + \partial_{x_i}[A_i(w)](x, t) = S(x, t, w) \qquad \text{in} \quad \Omega \times (0, T) \tag{70}$$

$$\text{Boundary conditions for } w \text{ on } \Gamma_0 \times (0, T) \text{ and } \Gamma_1 \times (0, T) \tag{71}$$

$$w(x, 0) = w^0(x) \qquad \text{in } \Omega \tag{72}$$

Here, $S(x, t, .)$ is a source term, which is in $L^\infty(\Omega \times (0, T); C^1)$ and satisfies $S(, x, t, 0) \equiv 0$. As before $w_\epsilon(x, t) = \int_V g_\epsilon(x, v, t)\, dv$ and $\chi_w$ is defined by the relation (8).

In the full space case $\Omega = \mathbb{R}^d$, a brief study of the inhomogeneous scalar conservation laws above has been given in ref. 7 in connection with the kinetic model

$$[\partial_t + a(v)\cdot\partial_x]\, g_\epsilon(x, v, t)$$
$$= \frac{1}{\epsilon}\,(\chi_{w_\epsilon(x,t)}(v) - g_\epsilon(x, v, t))$$
$$+ S'(x, t, v)\, g_\epsilon(x, v, t) \qquad \text{in} \quad \Omega\times V\times(0, T) \tag{73}$$
$$g_\epsilon(x, v, t) = g_{\epsilon 0}(x, v, t) \qquad \text{on} \quad \Gamma_0^- \times (0, T) \tag{74}$$
$$g_\epsilon(x, v, t) = g_{\epsilon 1}(x, v, t) \qquad \text{on} \quad \Gamma_1^- \times (0, T), \tag{75}$$
$$g_\epsilon(x, v, 0) = g_\epsilon^0(x, v) \qquad \text{in} \quad \Omega\times V \tag{76}$$

As compared with the kinetic model (73)–(76) proposed in ref. 7, our kinetic model (66)–(69) is more appropriate to describe the physics at the microscopic level, which yields the conservation laws (70)–(72) at the macroscopic level as the miscropscopic scale tends to 0. Its analysis does not require additional assymptions on the source terms as in ref. 7. We shall clarify this later.

Since our kinetic model is new, we shall also indicate how our analysis extend to the full space case i.e., $\Omega = \mathbb{R}^d$.

We begin with an existence and uniqueness result for the kinetic model.

**Theorem 5.1.** Assume that

$$g_\epsilon^0 \in L^1(\Omega\times V),\ \ a(v)\cdot n g_{\epsilon 1} \in L^1(\Gamma_1^- \times (0, T)),\ \ a(v)\cdot n g_{\epsilon 0} \in L^1(\Gamma_0^- \times (0, T))$$

Then the kinetic model (66)–(69) has a unique solution in $L^\infty([0, T]; L^1(\Omega\times V))$. Moreover, $g_\epsilon$ satisfies the integral representation

In $\Omega_0$

$$g_\epsilon(x, v, t) = g_\epsilon\left(\left(0, x_\star - \frac{x_1}{a_1(v)}\, a_\star(v)\right), v - \frac{x_1}{a_1(v)}\, S(x, t, v), t - \frac{x_1}{a_1(v)}\right)$$
$$\times \exp\left(-\frac{x_1}{\epsilon a_1(v)}\right) + \frac{1}{\epsilon}\int_{t-\frac{x_1}{a_1(v)}}^{t} \exp((s-t)/\epsilon)\, \chi_{w_\epsilon(x(s), s)}(v(s))\, ds$$

In $\Omega_{01}$

$$g_\epsilon(x, v, t) = g_\epsilon^0(x - a(v)\, t, v - tS(x, t, v)) \exp(-t/\epsilon)$$
$$+ \frac{1}{\epsilon}\int_0^t \exp((s-t)/\epsilon)\, \chi_{w_\epsilon(x(s), s)}(v(s))\, ds$$

In $\Omega_1$

$$g_\epsilon(x,v,t)=g_{\epsilon 1}\left(\left(1,x_\star+\frac{1-x_1}{a_1(v)}\,a_\star(v)\right),v+\frac{1-x_1}{a_1(v)}\,S(x,t,v),t-\frac{x_1-1}{a_1(v)}\right)$$
$$\times\exp\left(\frac{1-x_1}{\epsilon a_1(v)}\right)+\frac{1}{\epsilon}\int_{t-\frac{x_1-1}{a_1(v)}}^{t}\exp((s-t)/\epsilon)\,\chi_{w_\epsilon(x(s),s)}(v(s))\,ds$$

where $x(s)=x+(s-t)\,a(v)$, $x=(x_1,x_\star)$, $a(v)=(a_1(v),a_\star(v))$, and $v(s)=v+(s-t)\,S(x,t,v)$.

Finally, let $g_\epsilon$ and $G_\epsilon$ be two solutions of (1)–(4) with corresponding densities $w_\epsilon(x,t)=\int_V g_\epsilon(x,v,t)\,dv$ and $W_\epsilon(x,t)=\int_V G_\epsilon(x,v,t)\,dv$; and let $g_\epsilon^0$, $g_{\epsilon 0}$, $g_{\epsilon 1}$ resp. $G_\epsilon^0$, $G_{\epsilon 0}$, $G_{\epsilon 1}$ denote the corresponding data. Let

$$S'_\infty(t)=\{\max_{x,v}S'(x,t,v):v\in\operatorname{supp}_v g_\epsilon(x,v,t)\cup\operatorname{supp}_v G_\epsilon(x,v,t)\}$$

We have

$$\begin{aligned}
&\|g_\epsilon-G_\epsilon\|_{L^\infty([0,T];\,L^1(\Omega\times V))}\\
&\quad\leqslant\exp\left(\int_0^T|S'_\infty(s)|\,ds\right)\\
&\quad\quad\times[\|g_\epsilon^0-G_\epsilon^0\|_{L^1(\Omega\times V)}+\|a(v)\cdot n(g_{\epsilon 0}-G_{\epsilon 0})\|_{L^1(\Gamma_0^-\times(0,T))}\\
&\quad\quad+\|a(v)\cdot n(g_{\epsilon 1}-G_{\epsilon 1})\|_{L^1(\Gamma_1^-\times(0,T))}]
\end{aligned}\tag{77}$$

$$\begin{aligned}
&\|g_\epsilon-G_\epsilon\|_{L^\infty([0,T];\,L^1(\Omega\times V))}+\|a(v)\cdot n(g_{\epsilon 0}-G_{\epsilon 0})\|_{L^1(\Gamma_0^+\times(0,T))}\\
&\quad\quad+\|a(v)\cdot n(g_{\epsilon 1}-G_{\epsilon 1})\|_{L^1(\Gamma_1^+\times(0,T))}\\
&\quad\leqslant\left[1+\int_0^t|S'_\infty(s)|\exp\left(\int_0^\sigma|S'_\infty(\sigma)|\,d\sigma\right)ds\right][\|g_\epsilon^0-G_\epsilon^0\|_{L^1(\Omega\times V)}\\
&\quad\quad+\|a(v)\cdot n(g_{\epsilon 0}-G_{\epsilon 0})\|_{L^1(\Gamma_0^-\times(0,T))}+\|a(v)\cdot n(g_{\epsilon 1}-G_{\epsilon 1})\|_{L^1(\Gamma_1^-\times(0,T))}]
\end{aligned}\tag{78}$$

The proof of this theorem follows by arguing along the lines of the proof of Theorem 2.1, with obvious modification to account for the source term. We only point out here how to integrate by part in the term $S\,\partial_v g_\epsilon$. Let $\varphi$ be as in the proof of Theorem 2.1. Let $\eta\in C_0^\infty(V)$ satisfy $0\leqslant\eta\leqslant 1$, $\eta\equiv 1$ on $[-1,1]$, and $\operatorname{supp}\eta\subset[-2,2]$. Let $\eta_n=\eta(v/n)$. After multiplying the equation for $g_\epsilon$ by $\varphi\eta_n$, the contribution of the source term is

$$\begin{aligned}
&\int_{\Omega\times V\times(0,t)}S(x,t,v)\,\partial_v g_\epsilon\varphi\eta_n\\
&\quad=-\int_{\Omega\times V\times(0,t)}g_\epsilon\,\partial_v S(x,t,v)\,\varphi\eta_n-g_\epsilon S(x,t,v)\,\partial_v\varphi\eta_n-g_\epsilon S(x,t,v)\,\varphi\,\partial_v\eta_n
\end{aligned}\tag{79}$$

After passing to the limit as $n \to \infty$, the right hand side converges to

$$-\int_{\Omega \times V \times (0,t)} g_\epsilon\, \partial_v S(x,t,v)\, \varphi - g_\epsilon S(x,t,v)\, \partial_v \varphi$$

We also pass to the limit as $n \to \infty$ in the other terms. The rest of the proof proceeds as in the proof of Theorem 2.1 with appropriate modifications due to the source term.

We shall give below an entropy inequality for the solution of the kinetic problem. This is stated in the following theorem.

**Theorem 5.2.** The solution to the kinetic problem satisfies the relation

$$\begin{aligned}
&-\int_{\Omega \times V \times (0,T)} (\partial_t + a(v)\cdot\partial_x)(\psi)\, |g_\epsilon - \chi_k| + \int_{\Gamma_0^- \times (0,T)} a(v)\cdot n\psi\, |g_{\epsilon 0} - \chi_k| \\
&\quad + \int_{\Gamma_1^- \times (0,T)} a(v)\cdot n\psi\, |g_{\epsilon 1} - \chi_k| \\
&\quad \leqslant \int_{\Omega \times V \times (0,T)} g_\epsilon \psi\, \partial_v S\, \mathrm{sign}(g_\epsilon - \chi_k) \\
&\qquad \forall \psi \in C_0^1(\bar{\Omega} \times V \times (0,T)), \quad \psi \geqslant 0, \quad \forall k \in \mathbb{R}
\end{aligned} \tag{80}$$

Before we state our main convergence results, we shall give below a definition of a solution to the conservation laws with source term (70)–(72). This definition selects a physically correct solution to this problem.

**Definition 5.1.** We say that $w \in BV_{\mathrm{loc}}(\Omega \times (0,T)) \cap L^\infty(\Omega \times [0,T])$ is a weak entropic solution of the problem (70)–(72) if we have

$$\begin{aligned}
&-\int_{\Omega \times (0,T)} (|w-k|\, \partial_t \psi + \mathrm{sign}(w-k)(A(w)-A(k))\cdot \nabla_x \psi) \\
&\quad + \int_{\Gamma_1 \times (0,T)} \psi\, \mathrm{sign}(w_1 - k)((A(w_1)\cdot n)^- - (A(k)\cdot n)^-) \\
&\quad + \int_{\Gamma_0^- \times (0,T)} a(v)\cdot n\psi\, |g_0 - \chi_k| \\
&\quad \leqslant \int_{\Omega \times (0,T)} \psi S(x,t,w)\, \mathrm{sign}(w-k) \\
&\qquad \forall \psi \in C_0^1(\bar{\Omega} \times V \times (0,T)), \quad \psi \geqslant 0, \quad \forall k \in \mathbb{R}
\end{aligned}$$

and $w$ satisfies the initial condition

$$w(x,0)=w^0(x) \qquad \text{in } \Omega$$

We mention here that the kinetic entropy relations given in ref. 7 on p. 516, formula (5.5) for the kinetic model (73)–(76) and their corresponding macroscopic "continuum limit" entropy inequality given at the end of p. 516 in ref. 7 for the conservation laws with source terms (70)–(72) are not correct.

Next we shall state the main convergence results about the kinetic distributions and their moments for the source case.

**Theorem 5.3.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^-\times[0,T])}<C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega\times V)}<C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^-\times[0,T])}<C_3,$$
$$\|g_\epsilon^0\|_{L^1(\Omega\times V)}<C_4, \qquad \|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^-\times(0,T))}<C_5,$$
$$\|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^-\times(0,T))}<C_6, \qquad \|g_\epsilon^0\|_{L^1(V;\,BV_{\mathrm{loc}}(\Omega))}<C_7$$

with $C_i$, $i=1,\ldots,7$ positive constants independent of $\epsilon$.

Assume also that the initial and boundary data $f_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v\in V$ with supports included in a fixed compact set independent of $\epsilon$. Finally assume that as $\epsilon\to 0$,

$$\|w_\epsilon(\cdot,0)-w^0(\cdot)\|_{L^1_{\mathrm{loc}}(\Omega)}=\left\|\int_V g_\epsilon^0(\cdot,v)-w^0(\cdot)\right\|_{L^1_{\mathrm{loc}}(\Omega)}\to 0 \tag{81}$$

$$a(v)\cdot n g_{\epsilon 0}\to a(v)\cdot n g_0 \qquad \text{strongly in} \quad L^1(\Gamma_0^-\times(0,T)) \tag{82}$$

$$a(v)\cdot n g_{\epsilon 1}\to a(v)\cdot n g_1=a(v)\cdot n\chi_{w_1} \qquad \text{strongly in} \quad L^1(\Gamma_1^-\times(0,T)) \tag{83}$$

Then $w_\epsilon$ converges strongly in $L^1(\Omega\times V\times(0,T))$,as $\epsilon$ goes to 0, to an entropic solution of the problem (5)–(7) in the sense of Definition 5.1.

The theorem above does not provide a strong convergence uniform in $\epsilon$ and time of the density distribution to the equilibrium distribution. This is due to the presence of initial layers and the lack of the control of the velocity variation of the density distribution. Under the present assumptions (assumptions of Theorem 5.3) only a uniform control of the spatial variation on the microscopic scale and a uniform control of the temporal variation only at the macroscopic level are allowed (consult Lemma 2.4(1) and (2) and the remark after the proof of the corresponding Theorem in the sourceless case). The uniform control of the temporal variation of the kinetic distribution can be achieved only if we can control uniformly, in addition to the spatial variation, the velocity variation and the initial temporal variation of the kinetic distribution. For this purpose we assume that the kinetic initial data satisfies

$$\|g_\epsilon^0(\cdot,\cdot)-\chi_{w^0(\cdot)}(\cdot)\|_{L^1_{\text{loc}}(\Omega\times L^1(V))}\xrightarrow[\epsilon\to 0]{} 0$$
$$\|g_\epsilon^0\|_{L^1_{\text{loc}}(\Omega;\, B(V))}<C$$

Under the new additional assumptions, we obtain the following uniform in $\epsilon$ and time convergence of the kinetic ditribution to an equilibrium distribution.

**Theorem 5.4.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^-\times[0,T])}<C_1,\quad \|g_\epsilon^0\|_{L^\infty(\Omega\times V)}<C_2,\quad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^-\times[0,T])}<C_3,$$
$$\|g_\epsilon^0\|_{L^1(\Omega\times V)}<C_4,\quad \|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^-\times(0,T))}<C_5,\quad \|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^-\times(0,T))}<C_6$$
$$\|g_\epsilon^0\|_{L^1(V;\, BV_{\text{loc}}(\Omega))}<C_7,\quad \|g_\epsilon^0\|_{L^1_{\text{loc}}(\Omega;\, BV(V))}<C_8$$

with $C_i$, $i=1,\dots,8$ positive constants independent of $\epsilon$.

Assume also that the initial and boundary data $f_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v\in V$ with supports included in a fixed compact set independent of $\epsilon$. Finally assume that as $\epsilon\to 0$,

$$\|g_\epsilon^0(\cdot,\cdot)-\chi_{w^0(\cdot)}(\cdot)\|_{L^1_{\text{loc}}(\Omega\times L^1(V))}\xrightarrow[\epsilon\to 0]{} 0 \tag{84}$$
$$a(v)\cdot n g_{\epsilon 0}\to a(v)\cdot n g_0 \qquad \text{strongly in}\quad L^1(\Gamma_0^-\times(0,T)) \tag{85}$$
$$a(v)\cdot n g_{\epsilon 1}\to a(v)\cdot n g_1 = a(v)\cdot n\chi_{w_1} \qquad \text{strongly in}\quad L^1(\Gamma_1^-\times(0,T)) \tag{86}$$

Then $g_\epsilon$ converges strongly in $L^\infty([0,T];\, L^1(\Omega\times V))$, as $\epsilon$ goes to 0, to $\chi_w$ and $w$ is an entropic solution of the problem (5)–(7) in the sense of Definition 5.1.

**Remark 5.1.** (1) Remark 3.1 is also valid for Theorems 5.3 and 5.4.

(2) Notice that Theorems 5.3 and 5.4 are also valid for the simpler case of full space $\Omega=\mathbb{R}^d$ with appropriate modifications. We shall compare below our results for the full space case to those of ref. 7. For our generalized kinetic model the corresponding theorem to Theorem 5.3 for the full space case is obtained under no additional assumptions on the data or source terms. The analysis in ref. 7 required the additional assumption that the source terms are in $BV(\Omega)$. However, to obtain the uniform in $\epsilon$ and time convergence of the density distribution to an equilibrium distribution (the corresponding theorem to Theorem 5.4 for the full space case), we had

to assume an additional assumption that the initial ditribution $g_\epsilon^0$ is uniformly bounded in $L^1_{\mathrm{loc}}(\Omega; BV(V))$. As a result in our case the existence theory for conservation laws with source terms is obtained under no additional assumptions on the source terms as opposed to the existence theory given in ref. 7 which required the additional assumption that the source terms are $BV$. Thus our theory is more general.

To prove these theorems we argue along the lines of the proof of Theorem 3.1 for the sourceless case, with appropriate modifications due to the source term. We shall therefore state without proofs the corresponding lemmas with the necessary modifications due to the presence of the source term.

We begin with $L^\infty$ estimates.

**Lemma 5.1.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega\times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3$$

with $C_1$, $C_2$, and $C_3$ positive constants independent of $\epsilon$. Then $g_\epsilon$ is uniformly bounded in $L^\infty(\Omega\times V\times[0,T])$. Moreover we have

$$\|g_\epsilon\|_\infty \leqslant [\max(\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^-\times[0,T])}, \|g_\epsilon^0\|_{L^\infty(\Omega\times V)}, \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^-\times[0,T])})+1] \times \exp\left(\int_0^T |S'_\infty(\tau)|\, d\tau\right)$$

Here

$$S'_\infty(t) = \{\max_{x,v} S'(x,t,v) : v \in \mathrm{supp}_v\, g_\epsilon(x,v,t)\}$$

**Lemma 5.2.** Assume that

$$\|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^-\times(0,T))} < C_1, \qquad \|g_\epsilon^0\|_{L^1(\Omega\times V)} < C_2,$$
$$\|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^-\times(0,T))} < C_3$$

with $C_1$, $C_2$, and $C_3$ positive constants independent of $\epsilon$. Then $g_\epsilon$ is uniformly bounded in $L^\infty([0,T]; L^1(\Omega\times V))$ and $w_\epsilon$ is uniformly bounded in $L^\infty([0,T]; L^1(\Omega))$. Moreover, we have

$$\begin{aligned}\|w_\epsilon\|_{L^\infty([0,T];L^1(\Omega))} &\leqslant \|g_\epsilon\|_{L^\infty([0,T];L^1(\Omega\times V))}\\ &\leqslant \exp\left(\int_0^T |S'_\infty(\tau)|\,d\tau\right)[\|a(v)\cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^-\times(0,T))}\\ &\quad + \|a(v)\cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^-\times(0,T))} + \|g_\epsilon^0\|_{L^1(\Omega\times V)}]\end{aligned}$$

**Lemma 5.3.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega \times V)} < C_2,$$
$$\|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3$$

with $C_1$, $C_2$, and $C_3$ positive constants independent of $\epsilon$. Assume also that the initial and boundary data $g_\epsilon^0$, $g_{\epsilon 0}$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$. Then

(i) $w_\epsilon$ is uniformly bounded in $L^\infty(\Omega \times [0, T])$.

(ii) $g_\epsilon$ remains compactly supported in $v \in V$ with support included in a fixed compact set independent of $\epsilon$.

(iii) The speed of propagation $a(v)$ is finite.

**Lemma 5.4.** Assume that

$$\|g_{\epsilon 0}\|_{L^\infty(\Gamma_0^- \times [0,T])} < C_1, \qquad \|g_\epsilon^0\|_{L^\infty(\Omega \times V)} < C_2, \qquad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_3,$$
$$\|g_\epsilon^0\|_{L^1(\Omega \times V)} < C_4, \qquad \|a(v) \cdot n g_{\epsilon 0}\|_{L^1(\Gamma_0^- \times (0,T))} < C_5,$$
$$\|a(v) \cdot n g_{\epsilon 1}\|_{L^1(\Gamma_1^- \times (0,T))} < C_6, \qquad \|g_\epsilon^0\|_{L^1(V; BV_{\mathrm{loc}}(\Omega))} < C_7,$$

with $C_i$, $i = 1, \ldots, 7$ positive constants independent of $\epsilon$. Assume also that the initial and boundary data $f_{\epsilon 0}$, $g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of $\epsilon$.

Then

(1) $g_\epsilon(\cdot, \cdot, t)$ and $w_\epsilon(\cdot, t)$, $t \in [0, T]$ are uniformly bounded in $BV_{\mathrm{loc}}(\Omega \times L^1(V))$ and $BV_{\mathrm{loc}}(\Omega)$ respectively. More precisely, if $U$ and $O$ are open bounded subsets of $\Omega$ such that $\bar{U} \subset O \subset \bar{O} \subset \Omega$, we have for $i = 1, \ldots, d$

$$\int_{U \times V} |\tau_h^i g_\epsilon - g_\epsilon| \leqslant \exp\left(\int_0^t |S'_\infty(s)|\, ds\right) \int_{O \times V} |\tau_h^i g_\epsilon^0 - g_\epsilon^0|$$

(2) $w_\epsilon$ is time Lipschitz continuous in $L^1_{\mathrm{loc}}(\Omega)$ uniformly in $\epsilon$; i.e., for any open bounded subset $U$ of $\Omega$ with $\bar{U} \subset \Omega$, we have

$$\begin{aligned}
&\|w_\epsilon(\cdot, t_2) - w_\epsilon(\cdot, t_1)\|_{L^1(U)} \\
&\quad < (a_\infty \|g_\epsilon\|_{L^\infty([0,T]; BV(U \times L^1(V)))} + \|\partial_v S\|_{L^\infty(\Omega \times [0,T])} \|g_\epsilon\|_{L^\infty(\Omega \times V \times [0,T])})(t_2 - t_1) \\
&\quad < C(t_2 - t_1), \qquad \forall 0 \leqslant t_1 < t_2 \leqslant T
\end{aligned} \tag{87}$$

where $C$ is a constant depending on $U$ but is independent of $\epsilon$ and $a_\infty$ is introduced in the proof of Lemma 2.3 above.

(3) Under the additional assumptions

$$\|g_\epsilon^0(\cdot,\cdot)-\chi_{w^0(\cdot)}(\cdot)\|_{L^1_{\mathrm{loc}}(\Omega\times L^1(V))} \xrightarrow[\epsilon\to 0]{} 0$$

$$\|g_\epsilon^0\|_{L^1_{\mathrm{loc}}(\Omega;\, BV(V))} < C_8,$$

$g_\epsilon(\cdot,\cdot,t)$, $t\in[0,T]$ is uniformly bounded in $BV_{\mathrm{loc}}(V; L^1_{\mathrm{loc}}(\Omega))$. Moreover, we can estimate the error between the kinetic solution and exact entropy solution as follows

$$\begin{aligned}
&\|g_\epsilon-\chi_{w_\epsilon}\|_{L^\infty([0,T];\, L^1_{\mathrm{loc}}(\Omega\times L^1(V)))}\\
&\quad\leqslant \epsilon a_\infty \,\|g_\epsilon^0(x,v)\|_{BV_{\mathrm{loc}}(\Omega\times L^1(V))}+\epsilon a_\infty\,\|g_\epsilon(x,v,t)\|_{L^\infty([0,T];\, BV_{\mathrm{loc}}(\Omega\times L^1(V)))}\\
&\qquad+2\,\|g_\epsilon^0(x,v)-\chi_{w^0(x)}\|_{L^1_{\mathrm{loc}}(\Omega\times L^1(V))}+\epsilon\,\max_v\,\|S\|_{L^\infty(\Omega\times[0,T])}\\
&\qquad(\|g_\epsilon(x,v,t=0)\|_{BV(V\times L^1(O))}+\|g_\epsilon(x,v,t)\|_{BV(V\times L^1(O))})\\
&\qquad\xrightarrow[\epsilon\to 0]{} 0
\end{aligned}\tag{88}$$

(4) The function $w_\epsilon$ is uniformly bounded in $BV_{\mathrm{loc}}(\Omega\times(0,T))$.

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